Machine Learning in Finance

Exercise sheet 10

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$.

Exercise 10.1 (Signatures and reservoirs computing)

(a) Let $X \in \mathcal{C}_0^1([0,T], \mathbb{R}^n)$ satisfying the dynamic:

$$dX_t = \sum_{k=1}^{m} V_k(X_t) dW_t^k, \quad X_t \in \mathbb{R}^n, W_t \in \mathbb{R}^m, V^k : \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

where $(W_t)_{t=0}^{\infty}$ is a Brownian motion. Prove that

$$X_t = \sum_{d=0}^{\infty} \sum_{i_1, \cdots, i_d = 1}^{n} \left(\int_{0 \le t_1 \le \cdots \le t_d \le t} dW_{t_1}^{i_1} \cdots dW_{t_d}^{i_d} \right) V^{i_d} \cdots V^{i_1}(X_0) \cdot X_0. \tag{2}$$

where

$$Vf(x) = df(x) \cdot V(x).$$

(b) Rewrite (2) with signature in the form of the following:

$$X_t = \langle \mathbf{R}, \mathbf{Sig}_{[0,t]}(W) \rangle X_0, \tag{3}$$

and express the readout $\mathbf{R}$ with $(V^k)_{k=1}^m$ (notice that $\mathbf{R}$ depends on X_0).

(c) Relate (3) with reservoirs computing.

Exercise 10.2 (Randomized signature)

(a) Prove the Johnson-Lindenstrauss Lemma (see lecture notes): For any $0 < \epsilon < 1$ and any integer n, let k be a positive integer such that

$$k \ge \frac{24}{3\epsilon^2 - 2\epsilon^3} \log n$$

then for any set A of at most n points in $\mathbb{R}^d$, there exists $f : \mathbb{R}^d \to \mathbb{R}^k$ s.t. for all $x, y \in A$ that

$$(1-\epsilon)\|x-y\|^2 \le \|f(x) - f(y)\|^2 \le (1+\epsilon)\|x-y\|^2. \tag{4}$$

Johnson-Lindenstrauss Lemma not only prove the existence but also provide the projection, what is the projection considered in the proof of Johnson-Lindenstrauss Lemma?

(b) Let $X \in \mathcal{C}_0^1([0,T], E)$ and define $\pi_n : \mathbf{T}(E) \to \mathbf{T}^{(n)}(E)$ the projection such that for all $\mathbf{x} \in \mathbf{T}(E)$

$$\pi_n\big((\mathbf{x}_k)_{k \ge 0}\big) = (\mathbf{x}_k)_{k \le n} \tag{5}$$

then $S_t = \mathbf{Sig}_{[0,t]}^{(n)}(X)$ satisfies for all $t \in [0,T]$ that

$$dS_t = \pi_n(S_t \otimes dX_t), \quad S_0 = (1, 0, \cdots). \tag{6}$$

From this subproblem we can actually see signature as a solution of ODE which closely relates to the definition of randomized signature.

(c) Recall that randomized signature is defined as the solution of the following dynamic:

$$dRS_t = \sum_{i=1}^{d} \sigma(A_i RS_t + b_i) dX_t^i, \quad RS_0 \in \mathbb{R}^k, \tag{7}$$

where A_i and b_i are pre-randomized and σ be a non-linear function. Let σ be real analytic with infinite radius of convergence and let $A_1, \ldots, A_d$ and $b_1, \ldots, b_d$ be independent samples of a probability law absolutely continuous with respect to Lebesgue measure. Then prove that linear combinations of randomized signature for all possible initial values $RS_0 \in \mathbb{R}$ are dense in $C(K)$ on compact subsets of bounded variation curves.

References

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