Machine Learning in Finance

Exercise sheet 2

Exercise 2.1 (Stone-Weierstrass theorem [1])

(a) Construct a sequence of polynomials converges pointwisely but not uniformly on $[0, 1]$.

(b) Construct a sequence of polynomials converges uniformly to $x \mapsto |x|$ on $[-1, 1]$. (Hint: Corollary 2.3. in [1])

(c) Prove that ReLU can be approximated uniformly by polynomials on $[-1, 1]$.

(d) Use the universal approximation theory of shallow neural networks on $[0, 1]$ to prove the Stone-Weierstrass theorem.

Exercise 2.2 (Bernstein approximation [5]) Let B_n^f be the n-th Bernstein approximation of $f \in \mathcal{C}^k([0, 1])$ where $k \in \mathbb{N}$.

(a) Compute the derivative of B_n^f. (Hint: Theorem 7.1.2 in [5])

(b) Prove that B_n^f converges to f in $\mathcal{C}^k([0, 1])$. (Hint: Theorem 7.1.6 in [5])

(c) Construct the Wiener measure with Bernstein polynomials in Levy's construction (as in Wendelin Werner's lecture on Brownian motion with piecewise linear functions). (Hint: Theorem 4.3 in [3])

Exercise 2.3 (Networks on discrete path spaces)

(a) Describe the space of paths $\omega : \{1, \dots, T\} \to \mathbb{R}^d$ as $\mathbb{R}^{dT}$.

(b) Describe a shallow neural network, whose value at time t depends on path. Formulate a universal approximation theorem in this setting.

Exercise 2.4 Code the Bernstein approximation of continuous functions.

References

[1] SAMEER CHAVAN. Problems and notes: Uniform convergence and polynomial approximation.

[2] Hassan Ismail Fawaz, Germain Forestier, Jonathan Weber, Lhassane Idoumghar, and Pierre-Alain Muller. Deep learning for time series classification: a review. *Data mining and knowledge discovery*, 33(4):917–963, 2019.

[3] Emmanuel Kowalski. Bernstein polynomials and Brownian motion. *The American Mathematical Monthly*, 113(10):865–886, 2006.

[4] George G Lorentz. *Bernstein polynomials*. American Mathematical Soc., 2013.

[5] George M Phillips. *Interpolation and approximation by polynomials*, volume 14. Springer Science & Business Media, 2003.