Machine Learning in Finance

Exercise sheet 8

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J \colon \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0,s], E)$ and $Y \in \mathcal{C}_0^1([s,t], E)$. The concatenated path $X \star Y \in \mathcal{C}_0^1([0,t], E)$ is defined by

$$(X \star Y)_u = \begin{cases} X_u & u \in [0,s] \\ Y_u + (X_s - Y_s) & u \in [s,t] \end{cases} \tag{1}$$

Exercise 8.1 (Basic properties of signatures)

(a) Let $X_t = t\mathbf{x} \in \mathbb{R}^d$ and prove that

$$\mathbf{Sig}_{[0,1]}(X) = (1, \mathbf{x}, \frac{\mathbf{x}^{\otimes 2}}{2!}, \cdots). \tag{2}$$

(b) Prove the Chen's identity:

$$\mathbf{Sig}_{[r,t]}(X \star Y) = \mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,t]}(Y). \tag{3}$$

(c) Prove the Chen's identity for truncated signature:

$$\mathbf{Sig}^{(M)}_{[r,t]}(X \star Y) = \mathbf{Sig}^{(M)}_{[r,s]}(X) \otimes \mathbf{Sig}^{(M)}_{[s,t]}(Y). \tag{4}$$

(d) Let X be linear on $[n, n+1]$ and let $X_{n+1} - X_n = \mathbf{x}_n$ for $n \in \mathbb{N}$, use Chen's identity to prove that

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \leq N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots). \tag{5}$$

Exercise 8.2 (Linear controlled ODE) Let $E = \mathbb{R}^d, W = \mathbb{R}^n$. Let $X \in \mathcal{C}_0^1([0,T], E)$ and let $B \colon E \to \mathbf{L}(W)$ be a bounded linear map. Consider

$$dY_t = B(dX_t)(Y_t) \tag{6}$$

If we denote $B^k = B(e_k)$, $k = 1, \cdots, d$ then

$$dY_t = \sum_{k=1}^{d} B^k(Y_t) dX_t^k. \tag{7}$$

Prove that

$$Y_t = \Big(\sum_{k=0}^{\infty} B^{\otimes k}\Big)\Big(\mathbf{Sig}_{[0,t]}(X)\Big) Y_0. \tag{8}$$

This implies that the solution of controlled SDE could be written as a linear function on signature stream of driving path. This implies that signature stream is a promising feature for controlled ODE.

Exercise 8.3 Code the signature of piecewise linear paths. See notebook.

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.