Machine Learning in Finance

Exercise sheet 9

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$.

Exercise 9.1 (Properties of the space of signatures)

(a) Let $X \in \mathcal{C}_0^1([S_1, T_1], E)$ and $\tau : [S_2, T_2] \to [S_1, T_1]$ a non-decreasing surjective reparametrization. Then

$$\mathbf{Sig}_{[S_2,T_2]}(X_{\tau(\cdot)}) = \mathbf{Sig}_{[S_1,T_1]}(X). \tag{1}$$

(b) Let $X \in \mathcal{C}_0^1([0, T], E)$ and $X_0 = 0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2. \tag{2}$$

(c) Prove that neither is $\mathbf{Sig}_{[0,1]}$ surjective nor the range of which a linear subspace of $\mathbf{T}(E)$.

(d) Prove that signature of the augmented paths i.e. $\bar{X}_t = (t, X_t)$ is unique.

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.