Machine Learning in Finance

Solution sheet 2

Exercise 2.1 (Stone-Weierstrass theorem [1])

(a) Construct a sequence of polynomials converges pointwisely but not uniformly on $[0,1]$.

(b) Construct a sequence of polynomials converges uniformly to $x \mapsto |x|$ on $[-1,1]$. (Hint: Corollary 2.3. in [1])

(c) Prove that ReLU can be approximated uniformly by polynomials on $[-1,1]$.

(d) Use the universal approximation theory of shallow neural networks on $[0,1]$ to prove the Stone-Weierstrass theorem.

Solution 2.1

(a) Consider the function $f_n(x) = nx(1-x^2)^n$ for $x \in [0,1]$.

(b) Consider the following map

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x)), \tag{1}$$

which is a contraction on $[0,1)$ and the special case $x=1$ is obvious.

(c) $g(x) = \frac{1}{2}(x + |x|)$

(d) Since ReLU can be approximated uniformly by polynomials on $[0,1]$, composition of affine function and ReLU can be uniformly by polynomials on $[0,1]$. Thus, shallow neural networks can be uniformly by polynomials on $[0,1]$. Therefore, by UAT, polynomials can uniformly approximate any continuous function on $[0,1]$.

Exercise 2.2 (Bernstein approximation [5]) Let B_n^f be the n-th Bernstein approximation of $f \in \mathcal{C}^k([0,1])$ where $k \in \mathbb{N}$.

(a) Compute the derivative of B_n^f. (Hint: Theorem 7.1.2 in [5])

(b) Prove that B_n^f converges to f in $\mathcal{C}^k([0,1])$. (Hint: Theorem 7.1.6 in [5])

(c) Construct the Wiener measure with Bernstein polynomials in Levy's construction (as in Wendelin Werner's lecture on Brownian motion with piecewise linear functions). (Hint: Theorem 4.3 in [3])

Solution 2.2

(a) See Theorem 7.1.2 in [5]

(b) See Theorem 7.1.6 in [5].

(c) See Theorem 4.3 in [3].

Exercise 2.3 (Networks on discrete path spaces)

(a) Describe the space of paths $\omega : \{1, \dots, T\} \to \mathbb{R}^d$ as $\mathbb{R}^{dT}$.

(b) Describe a shallow neural network, which depends on value at time t and on path information up to time t. Formulate a universal approximation theorem in this setting.

Solution 2.3

(a) Maps from $\{1, \dots, T\}$ to $\mathbb{R}^d$ expressed by $\mathbb{R}^{dT}$.

(b) A neural network with input space $\mathbb{R}^{dt}$ for fixed t, a neural network with input space at least $\mathbb{R}^{dT}$ (might be larger if allow duplicated information in input space). UAT for space space is concerning universal approximation of continuous functional on path spaces e.g. the running max of a discrete path.

Exercise 2.4 Code the Bernstein approximation of continuous functions.

Solution 2.4 See notebook.

References

[1] SAMEER CHAVAN. Problems and notes: Uniform convergence and polynomial approximation.

[2] Hassan Ismail Fawaz, Germain Forestier, Jonathan Weber, Lhassane Idoumghar, and Pierre-Alain Muller. Deep learning for time series classification: a review. *Data mining and knowledge discovery*, 33(4):917–963, 2019.

[3] Emmanuel Kowalski. Bernstein polynomials and brownian motion. *The American Mathematical Monthly*, 113(10):865–886, 2006.

[4] George G Lorentz. *Bernstein polynomials*. American Mathematical Soc., 2013.

[5] George M Phillips. *Interpolation and approximation by polynomials*, volume 14. Springer Science & Business Media, 2003.