Machine Learning in Finance

Solution sheet 6

Exercise 6.1 (Convex risk measure)

(a) Recall the definition of convex risk measure on L^∞. (Hint: monotone decreasing, convex, cash invariant)

(b) Assume $l\colon \mathbb{R} \to \mathbb{R}$ is a loss function i.e. continuous, non-decreasing and convex. We define

$$\rho(X) = \inf_{w \in \mathbb{R}} \{w + \mathbb{E}[l(-X - w)]\}. \tag{1}$$

Prove that ρ define a convex risk measure.

(c) Let $\lambda > 0$ and $l(x) = \exp(\lambda x) - \frac{1+\log(\lambda)}{\lambda}$. Solve (1) explicitly, write down the minimizer, and prove that

$$\rho(X) = \frac{1}{\lambda} \log \mathbb{E}[\exp(-\lambda X)] \tag{2}$$

which is the entropic risk measure.

(d) Let $\alpha \in (0,1)$, $l(x) = \frac{1}{1-\alpha} \max(x, 0)$. Solve (1) explicitly, write down the minimizer, and prove that

$$\rho(X) = \frac{1}{1-\alpha} \int_0^{1-\alpha} \mathrm{VaR}_\gamma(X) d\gamma \tag{3}$$

which is the expected shortfall. VaR is the value at risk.

(e) Is value at risk a convex risk measure on L^∞?

Solution 6.1

(a)
- If $X_1 \geq X_2$, then $\rho(X_1) \leq \rho(X_2)$.
- $\rho(\alpha X_1 + (1-\alpha) X_2) \leq \alpha \rho(X_1)(1-\alpha)\rho(X_2)$ for $\alpha \in [0,1]$
- $\rho(X + c) = \rho(X) - c$ for $c \in \mathbb{R}$

(b) See Lemma 3.7 in [1].

(c) See Example 3.8 in [1].

(d) See Example 3.9 in [1].

(e) No, value at risk is not a convex risk measure on L^∞, see [2].

Exercise 6.2 (Indifference pricing) Let ρ be a convex risk measure and

$$\pi(X) = \inf_{\delta \in \mathcal{H}} \rho(X + (\delta \cdot S)_T - C_T(\delta)) \tag{4}$$

(a) Recall the definition of indifference price p with π. Give financial interpretation of δ, $\mathcal{H}$, $(\delta \cdot S)$ and $C_T(\delta)$ above.

(b) Suppose $C_T = 0$ and Z is attainable by $Z = p^* + (\delta^* \cdot S)$. Prove that $p(Z) = p^*$.

(c) Suppose $C_T = 0$. Prove that the indifference price is bounded by super-hedging price.

Solution 6.2

(a) Indifference price:

$$p(Z) = \pi(-Z) - \pi(0). \tag{5}$$

δ is the trading strategy (holding). $\mathcal{H}$ is the set of admissible strategy. $(\delta \cdot S)$ is the wealth process by self-financing trading. $C_T(\delta)$ is the accumulated cost by trading strategy δ (also depending S).

(b) See Lemma 3.3 in [1].

(c) By monotonicity.

Exercise 6.3 (Path dependence hedging) Code a deep path dependent hedging strategy, see exercise notebook.

References

[1] Hans Buehler, Lukas Gonon, Josef Teichmann, and Ben Wood. Deep hedging. *Quantitative Finance*, 19(8):1271–1291, 2019.

[2] Hans Föllmer and Alexander Schied. *Stochastic finance*. de Gruyter, 2016.

[3] Blanka Horvath, Josef Teichmann, and Žan Žurič. Deep hedging under rough volatility. *Risks*, 9(7):138, 2021.