Modular Forms: Problem Sheet 10

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1. Let $\Gamma' \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups with $-I \in \Gamma'$. Suppose that $f \in \mathcal{S}_k(\Gamma) \subset \mathcal{S}_k(\Gamma')$ and that $g \in \mathcal{S}_k(\Gamma')$. Letting $\Gamma = \bigcup_i \alpha_i \Gamma'$, define the trace of g to be

$$\operatorname{trace} g = \sum_i g|_k \alpha_i \in \mathcal{S}_k(\Gamma).$$

   Show that

$$\langle f, g \rangle_{\Gamma'} = \langle f, \operatorname{trace} g \rangle_{\Gamma}.$$

2. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Show that if $\alpha^{-1}\Gamma\alpha = \Gamma$, then $|_k \alpha'$ is the adjoint operator of $|_k \alpha$ with respect to the Petersson inner product.

3. Define the normalized Petersson inner product as

$$[f, g]_\Gamma = \frac{1}{V_\Gamma} \langle f, g \rangle_\Gamma,$$

   for any two cusp forms $f, g \in \mathcal{S}_k(\Gamma)$, and with $V_\Gamma = \int_{D_\Gamma} \frac{dxdy}{y^2}$.

   i. Find a formula relating the volumes V_Γ and $V_{\mathrm{SL}_2(\mathbb{Z})}$ and the index $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$.

   ii. Show that if $\Gamma' \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ are congruence subgroups, then $[,]_\Gamma = [,]_{\Gamma'}$ on $\mathcal{S}_k(\Gamma)$.