Modular Forms: Problem Sheet 11

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1. (a) Let $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Show that $\gamma^{-1}\Gamma_1(N)\gamma = \Gamma_1(N)$, and so the operator $w_N = |_k\gamma$ is the double coset operator $[\Gamma_1(N)\gamma\Gamma_1(N)]_k$ on $\mathcal{S}_k(\Gamma_1(N))$. Show that $w_N \langle n \rangle w_N^{-1} = \langle n \rangle^*$ for all n such that $(n, N) = 1$, and thus for all n.

   (b) Let $\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N) = \cup_j \Gamma_1(N)\beta_j$ be a distinct union. From part (a), $\gamma\Gamma_1(N) = \Gamma_1(N)\gamma$ and similarly for γ^{-1}. Use this and the formula

   $$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \gamma^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \gamma$$

   to find coset representatives for $\Gamma_1(N)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_1(N)$. Use the formula for the adjoint computed in Theorem 3.4.6 and the coset representatives to show that

   $$T_p^* = w_N T_p w_N^{-1} \text{ and so } T_n^* = w_N T_n w_N^{-1} \text{ for all } n.$$

   (c) Show that $w_N^* = (-1)^k w_N$ and that $i^k w_N T_n$ is self-adjoint.

2. Let l be a prime and let $N \in \mathbb{Z}_{\geq 1}$. Recall the U_l and V_l operators and their action on the q-expansion of $f = \sum_n a_n q^n \in \mathcal{M}_k(\Gamma_1(N))$:

   $$U_l f = \sum_n a_{nl} q^n,$$
   $$V_l f = \sum_n a_n q^{nl}.$$

   Compute the composition $V_l \circ U_l$.

3. Recall that a lattice in $\mathbb{C}$ is a subgroup $\Lambda \subset \mathbb{C}$ of the form

   $$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

   for two $\mathbb{R}$-linearly independent complex numbers ω_1, ω_2. We make the normalizing convention that $\omega_1/\omega_2 \in \mathcal{H}$ and denote $\mathcal{L}$ the set of such lattices. Note that two lattices of $\mathcal{L}$, say $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $\Lambda' = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$, are equal if and only if there exists a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

   $$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \gamma \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

   For an arbitrary $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, we let $\tau = \omega_1/\omega_2 \in \mathcal{H}$ and $\Lambda_\tau := \tau\mathbb{Z} + \mathbb{Z}$. Then, the map

   $$\begin{array}{rccc} \varphi_\tau : & \mathbb{C}/\Lambda & \to & \mathbb{C}/\Lambda_\tau \\ & z + \Lambda & \mapsto & z/\omega_2 + \Lambda_\tau \end{array}$$

   is an isomorphism. Moreover, $\tau \in \mathcal{H}$ is uniquely determined up to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

   Finally, we say that a map

   $$\mathcal{F} : \mathcal{L} \to \mathbb{C}$$

is homogeneous of weight $k \in \mathbb{Z}$ if it satisfies

$$\mathcal{F}(\lambda\Lambda) = \lambda^{-k}\mathcal{F}(\Lambda) \text{ for all } \lambda \in \mathbb{C}^* \text{ and } \Lambda \in \mathcal{L}.$$

Note that the value of F at any $\Lambda \in \mathcal{L}$ is completely determined by its value at Λ_τ. To a homogeneous map, we associate a function $f : \mathcal{H} \to \mathbb{C}$ by letting $f(\tau) = \mathcal{F}(\Lambda_\tau)$ for all $\Lambda \in \mathcal{L}$. This function is then well-defined by the above paragraph.

(a) Show that such an f is modular, i.e. prove that $f(\gamma\tau) = \mathcal{F}(\Lambda_{\gamma\tau}) = j(\gamma,\tau)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

(b) For a prime number p, define the Hecke operator T_p by letting $T_p\mathcal{F}$ be the homogeneous function of weight k corresponding to $T_p f$ (where the second occurence of T_p is our usual Hecke operator on $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$). Show that

$$T_p\mathcal{F}(\Lambda) = \frac{1}{p} \sum_{\substack{\Lambda' \supset \Lambda \\ [\Lambda':\Lambda]=p}} \mathcal{F}(\Lambda'),\ \forall \Lambda \in \mathcal{L}.$$