Modular Forms: Problem Sheet 1

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1. Show that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$ is **transitive** (for any $z, z' \in \mathcal{H}$, there is $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g \circ z = z'$) and **faithful** (no non-identity element in $\mathrm{PSL}_2(\mathbb{R})$ acts trivially on $\mathcal{H}$).

2. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbb{Z})$. Show that if $c = 1$ and $d = 0$, and there is some $z \in D$ such that $\gamma z \in D$, then we must have one of the following possibilities:

   - $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and z is any element of D with $|z| = 1$,
   - $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $z = \gamma z = 1 + \rho$,
   - $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $z = \gamma z = \rho$.

   (This fills in a detail of Step 2 of Theorem 1.2.2.)

3. Express the element
$$\gamma = \begin{pmatrix} 8 & 29 \\ 11 & 40 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$
   in terms of the generators S and T.

4. Let $G = \mathrm{SL}_2(\mathbb{Z})$. Show that for $z \in D$, we have
$$G_z = \begin{cases} C_6 = \langle ST \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle & z = \rho \\ C_6 = \langle TS \rangle = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle & z = \rho + 1 \\ C_4 = \langle S \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle & z = i \\ C_2 = \langle -\mathrm{Id} \rangle = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle & z \notin \{i, \rho, \rho + 1\} \end{cases}$$