Modular Forms: Problem Sheet 2

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1. (a) Let a, b, c, d be integers. Show that the map $(m, n) \mapsto (ma + nc, mb + nd)$ is a bijection on $\mathbb{Z}^2 - \{(0, 0)\}$ if and only if $ad - bc = \pm 1$.

   (b) Hence show that for $k \geq 4$ even, the Eisenstein series

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k}$$

   satisfies $G_k|_k \gamma = G_k$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, without using Theorem 1.2.2 (a) as we did in class. (You may assume that the sum defining $G_k(z)$ converges absolutely for any $z \in \mathcal{H}$.)

2. (a) Prove the identity

$$\pi z \frac{\cos(\pi z)}{\sin(\pi z)} = \sum_{k \geq 0,\, \text{even}} (2\pi i)^k \frac{B_k}{k!} z^k \qquad \forall\, |z| < 1.$$

   (b) Show that

$$\pi z \frac{\cos(\pi z)}{\sin(\pi z)} = 1 - 2 \sum_{k \geq 2,\, \text{even}} \zeta(k) z^k \qquad \forall\, |z| < 1.$$

   (c) Deduce that the values of the Riemann zeta function at the even integers $k \geq 2$ are given by

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}$$

   (d) Prove that B_k is non-zero if and only if k is 1 or k is even.

3. (a) Show that the Eisenstein series E_4 has a simple zero at $z = \rho$ and no other zeros.

   (b) Similarly, show that E_6 has a simple zero at $z = i$ and no other zeros.