Modular Forms: Problem Sheet 3

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1. (a) Show that the Eisenstein series E_4 has a simple zero at $z = \rho$ and no other zeros.
   (b) Similarly, show that E_6 has a simple zero at $z = i$ and no other zeros.

2. Define the modular invariant $j(z) = \frac{E_4^3}{\Delta}$. *(This is an extremely important invariant, with applications to the elliptic curves and class field theory. The coefficients in its q-expansion are famous for their role in the* ***moonshine conjecture****, which links them to the representation theory of the monster group.)*
   (a) Prove that j is a weakly modular form of weight 0.
   (b) Show that j is holomophic on $\mathcal{H}$ and has a simple pole at ∞.
   (c) Show that j induces a bijection $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H} \cong \mathbb{C}$.

3. Show that for any $k \geq 0$, we have
$$|\{(a,b) \in \mathbb{Z}^2_{\geq 0} : 4a + 6b = k\}| = \dim M_k.$$

4. Let n be a positive integer.
   (a) Show (by quoting an appropriate theorem from your notes) that the dimension of M_{4n} is $1 + j$, where $j = \lfloor n/3 \rfloor$. Hence show that the functions
$$E_4^n, E_4^{n-3}\Delta, \ldots, E_4^{n-3j}\Delta^j$$
   are a basis of M_{4n}.
   (b) Let $M_{4n}(\mathbb{Z})$ denote the $\mathbb{Z}$-submodule of M_{4n} consisting of modular forms whose q-expansions have integer coefficients. Show that the above functions are a $\mathbb{Z}$-basis of $M_{4n}(\mathbb{Z})$. (You may assume that $\Delta \in M_{12}(\mathbb{Z})$.)