Modular Forms: Problem Sheet 4

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1. If f is a modular form (of any weight), write $a_i(f)$ for the coefficient of q^i in its q-expansion.
   (a) Find constants c_1 and c_2 such that $f = c_1 E_4^3 + c_2 E_6^2$ has $a_1(f) = 1$ and $a_2(f) = \sigma_{11}(2)$. Hence show that the constant γ_{12} such that
   $$E_{12} = 1 + \gamma_{12} \sum_{n \geq 1} \sigma_{11}(n) q^n$$
   is equal to $\frac{65520}{691}$.
   (b) Deduce that
   $$\zeta(12) = \frac{691}{638512875} \pi^{12}.$$
   (c) Find constants d_1, d_2 such that
   $$\Delta = d_1 E_{12} + d_2 E_4^3.$$
   Hence prove Ramanujan's congruence for the coefficients of Δ modulo 691, namely that
   $$\tau(n) = \sigma_{11}(n) \pmod{691}.$$

2. Let $N \geq 2$ and let $c, d \in \mathbb{Z}/N\mathbb{Z}$. We say that c and d are **coprime modulo** N if there is no $f \neq 0$ in $\mathbb{Z}/N\mathbb{Z}$ such that $fc = fd = 0$.
   (a) Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, then c and d are coprime modulo N.
   (b) Show that for any pair (c, d) that are coprime modulo N, there exist $c', d' \in \mathbb{Z}$ such that $c' = c$ and $d' = d \pmod{N}$ and $\mathrm{HCF}(c', d') = 1$.
   (c) Hence (or otherwise) show that the natural reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective for any $n \geq 2$.
   (d) Give an example of an integer N and an element of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ which is not the reduction of any element of $\mathrm{GL}_2(\mathbb{Z})$.

3. (a) Let D and N be positive integers, and let β be a (2x2) matrix with integral entries and determinant D. Prove that $\Gamma(DN) \subseteq \Gamma(N) \cap \beta^{-1}\Gamma(N)\beta$
   (b) Let Γ be any congruence subgroup, and let $\alpha \in \{A \in \mathrm{GL}_2(\mathbb{Q}) : \det(A) > 0\}$. Prove that the group $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$ is again a congruence subgroup.