Modular Forms: Problem Sheet 5

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1. Let Γ and Γ' be congruence subgroups such that $\Gamma' \trianglelefteq \Gamma$.
   1. Show that if $c, d \in C(\Gamma')$ are equivalent in $C(\Gamma)$, then $h_{\Gamma'}(c) = h_{\Gamma'}(d)$.
   2. Let $c \in \mathrm{Cusps}(\Gamma')$. Show that
   $$\sum_{\substack{d \in C(\Gamma') \\ d = c \text{ in } \mathrm{Cusps}(\Gamma)}} h_{\Gamma'}(d) = [\overline{\Gamma} : \overline{\Gamma'}] h_{\Gamma}(c).$$
   3. Hence show that for p odd, $\Gamma_1(p)$ has exactly $p-1$ cusps.
2. (a) Show that $\mathrm{SL}_2(\mathbb{Z})$ contains an index 2 subgroup Γ which is congruence of level 2.

   (b) Show that the only cusp of Γ is $[\infty]$. What is its width?
3. Show that the cusp $c = [1/2]$ of $\Gamma_1(4)$ is irregular, and find a generator of the corresponding subgroup H_c.
4. Let Γ and Γ' be congruence subgroups such that $\Gamma' \trianglelefteq \Gamma$. Let f be a meromorphic function on $\mathcal{H}$ that is weakly modular of weight k for Γ. Let $P' \in \mathrm{Cusps}(\Gamma')$, and let P be its image in $\mathrm{Cusps}(\Gamma)$. Then f is homomorphic at P if and only f (viewed as a weakly modular function of weight k for Γ') is holomorphic at P'. Also show that f vanishes at P if and only if f vanishes at P'.