Modular Forms: Problem Sheet 7

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1. Let $E_2(z)$ be as defined in Definition 1.7.1.
   (a) Show that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have
   $$(cz+d)^{-2}E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) - \frac{6ic}{\pi(cz+d)}.$$
   (b) For $N \geq 2$, define
   $$E_2^{(N)}(z) = E_2(z) - NE_2(Nz).$$
   Show that the function $E_2^{(N)}(z)$ is a modular form of weight 2 and level $\Gamma_0(N)$, and determine its values at the cusps.

2. Let Γ be a finite-index **even** subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and c a cusp of Γ. Assume that k is an even integer ≥ 4. Show that we have
   $$G_{k,\Gamma,c}(z) = \sum_{(m,n)\in S(c)} \frac{1}{(mz-n)^k}$$
   where $S(c)$ is the set of pairs $(m,n) \in \mathbb{Z}^2$ such that $\mathrm{HCF}(m,n) = 1$ and the element $\frac{n}{m} \in \mathbb{P}^1(\mathbb{Q})$ lies in the Γ-orbit c. Describe the sets $S(c)$ for each of the three cusps $\{\infty, 0, \frac{1}{2}\}$ of $\Gamma_1(4)$.

3. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$.. Decompose $\Gamma g \Gamma$ into left Γ-cosets, i.e. find $\{\alpha_i \in \mathrm{GL}_2(\mathbb{Q})^+\}$ such that
   $$\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma = \bigcup_i \Gamma\alpha_i.$$
   Hence show that if f is a modular form of weight k and leverl Γ, we have
   $$f|_k[\Gamma g \Gamma] = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + p^{k-1}f(pz)$$