# Modular Forms: Problem Sheet 8

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1. Find an integer $N$ and a Dirichlet character $\chi$ modulo $N$ such that
   (a) $\chi$ is primitive, but not injective;
   (b) $\chi$ is injective, but not primitive.

   Show that the situation of (b) can only occur if $N = 2 \bmod 4$.

2. Let $p$ and $q$ be distinct primes dividing the positive integer $N$. Show directly that if $\alpha_0, \dots, \alpha_{p-1}$ and $\beta_0, \dots, \beta_{q-1}$ are the left coset representatives for the double cosets $T_p$ and $T_q$ constructed in Proposition 3.2.2, then $\{\alpha_i \beta_j : 0 \le i \le p-1, 0 \le j \le q-1\}$ is a set of left coset representatives for the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix} \Gamma_1(N)$.

   Hence give a direct proof that $T_p T_q = T_q T_p$ in this case.

3. Suppose $p$ is a prime, $\Gamma = \Gamma_1(p)$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Find $p$ matrices $(g_j)_{j=0,\dots,p-1}$ in $\mathrm{GL}_2^+(\mathbf{Q})$ such that

$$\Gamma g \Gamma = \bigsqcup_{0 \le j < p} \Gamma g_j = \bigsqcup_{0 \le j < p} g_j \Gamma.$$