Modular Forms: Problem Sheet 9

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1. Prove the formula stated in Remark 3.2.29 of the lecture notes: let $f \in \mathcal{M}_k(\Gamma_1(N))$ with q-expansion

$$f(z) = \sum_{m=0}^{\infty} a_m(f) q^m, \ q = e^{2i\pi z},$$

and show that for all $n \in \mathbb{Z}^+$, $T_n(f)$ has Fourier expansion

$$\sum_{m=0}^{\infty} a_m(T_n f) q^m,$$

where

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(\langle d \rangle f).$$

Hint: *Since $\mathcal{M}_k(\Gamma_1(N))$ can be decomposed as a direct sum of eigenspaces of the form $\mathcal{M}_k(\Gamma_1(N), \chi)$, this is tantamount to showing that if $f \in \mathcal{M}_k(\Gamma_1(N), \chi)$, then*

$$a_m(T_n f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f), \ \forall n \in \mathbb{Z}^+.$$

2. Let $f \in \mathcal{M}_k(\Gamma_1(N), \chi)$ be a normalized Hecke eigenform and define its L-function to be the series

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f) n^{-s}.$$

This is a well-defined function when the real part of $s \in \mathbb{C}$ is large enough.
Express $L(f, s)$ as an Euler product, i.e. find an expression of $L(f, s)$ of the form

$$L(f, s) = \prod_{p \text{ prime}} L_p(f, s),$$

where $L_p(f, s)$ is a complex function that depends on p, s, and the Fourier coefficient $a_p(f)$.
Hint: *Use the intrinsic characterization of Hecke eigenforms proved in Proposition 3.2.34.*