

# **Modular Forms**

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# 0 Prologue

**Example 0.0.1.** Let  $z \in \mathbb{C}$ ,  $\Im(z) > 0$ . Let  $q = e^{2\pi iz}$  and define **Ramanujan's tau function**

$$\Delta(z) = q \cdot \prod_{n \in \mathbb{N}} (1 - q^n)^{24}.$$

This is one of the simplest examples of a modular form. Note that we can "multiply out" the product above which leads us to

$$\Delta(z) = \sum_{n \in \mathbb{N}} \tau(n) q^n$$

for some integers  $\tau(n)$ .

**Facts 0.0.2.**

- (1) Known to Weierstrass, 1850:

$$\Delta(z) = z^{-12} \cdot \Delta\left(-\frac{1}{z}\right)$$

- (2) Ramanujan proved in 1916 that the integers  $\tau(n)$  satisfy the equation

$$\tau(n) = \sum_{d|n} d^{11} \pmod{691}.$$

- (3) Ramanujan also conjectured  $\tau(nm) = \tau(n)\tau(m)$  for  $n, m$  coprime. This was proved by Mordell in 1917.

- (4) In 1972 Swinnerton-Dyer proved  $\tau(n)$  satisfies congruences like (2) modulo 2, 3, 5, 7, 23 and 691 but no other primes.

- (5) Ramanujan conjectured in 1916 for  $p$  prime holds  $|\tau(p)| < 2 p^{11/2}$ . This was proved in 1974 by Deligne.

- (6) The quantity

$$\frac{\tau(p)}{2p^{11/2}} \in [-1, 1]$$

is distributed in the interval  $[-1, 1]$  with density function proportional to  $\sqrt{1-x^2}$ . This was conjectured by Sato and Tate (1960s) and proved by Barnet-Lamb, Geraghty, Harris and Taylor in 2009 using Bau Chau Ngo's *Fundamental Lemma* which got Ngo the 2010 Fields Medal.

**Example 0.0.3.** We now consider another modular form

$$\begin{aligned} f(z) &= q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 + \dots \\ &= \sum_{n=1}^{\infty} a(n)q^n \quad \text{with } a(n) \in \mathbb{N} \end{aligned}$$

We will later prove the following results:

**Theorem.**

1. We have  $a(mn) = a(m)a(n)$  for all  $m, n \geq 1$  with  $(m, n) = 1$ .
2. We have  $|a(p)| \leq 2\sqrt{p}$  for all primes  $p$ .

It turns out that this modular form is closely related to the elliptic curve

$$E : Y^2 + Y = X^3 - X^2 - 10X - 20.$$

For  $p$  prime, denote by  $N(p)$  the number of points on the elliptic curve in  $\mathbb{F}_p$ . It is easy to see heuristically that  $N(p) \simeq p$ .

**Theorem.** (Hasse) We have

$$|p - N(p)| \leq 2\sqrt{p}.$$

The theory of modular forms allows one to prove that the elliptic curve  $E$  and the modular form  $f$  ‘correspond’ to each other in the following sense:

**Theorem.** For all primes  $p$ , we have

$$a(p) = p - N(p).$$

In particular, using the properties of the modular form  $f$ , we can easily calculate the quantity  $N(p)$  for all  $p$ , so  $f$  ‘knows’ about the behaviour of the elliptic curve over  $\mathbb{F}_p$ . We say that the elliptic curve  $E$  is **modular**. It is generally not too difficult to attach an elliptic curve to a modular form (this is called “Eichler–Shimura”); however, it is very difficult indeed to reverse this process, and this is the basis of Andrew Wiles’ work on Fermat’s Last Theorem. The proof of this result was later completed by Breuil–Conrad–Diamond–Taylor. I will talk a bit more about this when we discuss  $L$ -functions of modular forms.

# 1 The modular group

## 1.1 The upper half-plane

**Definition 1.1.1.** Let  $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$  the **upper half-plane**.

**Proposition 1.1.2.** The **special linear group**  $SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : \det(A) = 1\}$  acts on  $\mathcal{H}$  via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

*Proof.* For  $z \in \mathcal{H}$  is  $\Im(z) > 0$  and either  $c$  or  $d$  is nonzero, so  $cz + d \neq 0$ . Moreover

$$\Im\left(\frac{az + b}{cz + d}\right) = \frac{1}{|cz + d|^2} \Im((az + b)(c\bar{z} + d)).$$

Say  $z = x + iy$  for  $x, y \in \mathbb{R}$ .

$$\begin{aligned} \Im\left(\frac{az + b}{cz + d}\right) &= \frac{1}{|cz + d|^2} \Im\left(\underbrace{(ax + b)(cx + d) + acy^2}_{\in \mathbb{R}} + i \underbrace{(ad - bc)}_{=1} y\right) \\ &= \frac{1}{|cz + d|^2} \Im(z) > 0 \end{aligned}$$

Therefore  $\frac{az+b}{cz+d} \in \mathcal{H}$  for any  $z \in \mathcal{H}$ ,  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ .

Also it is easy to check that  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z = z$  and  $A(Bz) = (AB)z$  for any  $z \in \mathcal{H}$  and for any  $A, B \in SL_2(\mathbb{R})$ . Thus  $SL_2(\mathbb{R})$  acts on  $\mathcal{H}$ .  $\square$

**Note 1.1.3.** The matrix  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{R})$  acts trivially on  $\mathcal{H}$ , so the action of  $SL_2(\mathbb{R})$  on  $\mathcal{H}$  factors through the quotient  $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/(\pm 1)$ , the **projective special linear group**.

**Definition 1.1.4.** The *automorphy factor* is the function

$$j : SL_2(\mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C},$$

$$(g, z) \mapsto cz + d \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

**Proposition 1.1.5.** For any  $k \in \mathbb{Z}$ , we can define a right action of  $SL_2(\mathbb{R})$  on the set of holomorphic functions  $\mathcal{H} \rightarrow \mathbb{C}$  given by

$$(f|_k g)(z) := j(g, z)^{-k} f(gz)$$

where  $f : \mathcal{H} \rightarrow \mathbb{C}$  holomorphic,  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ . We will call this the **weight  $k$  action**.

*Proof.* Firstly we need to show that  $f|_k g$  is a well-defined holomorphic function  $\mathcal{H} \rightarrow \mathbb{C}$ . But this is obvious since  $cz + d \neq 0$  and  $gz \in \mathcal{H}$  for all  $z \in \mathcal{H}$ . Clearly also the equation  $f|_k 1 = f$  holds. Therefore it remains to show  $(f|_k g)|_k h = f|_k (gh)$  for arbitrary  $g, h \in \mathrm{SL}_2(\mathbb{R})$ . The left hand side of the equation can be rewritten as

$$\begin{aligned} (f|_k g)|_k h &= j(h, z)^{-k} ((f|_k g)(hz)) \\ &= j(h, z)^{-k} j(g, hz)^{-k} f(g(hz)) \end{aligned}$$

and the right hand side results in

$$f|_k (gh) = j(gh, z)^{-k} f((gh)z).$$

We already know  $(gh)z = g(hz)$ . So it remains to show  $j(gh, z) = j(h, z)j(g, hz)$ . This is the so called **cocycle relation** and can be checked easily.  $\square$

## 1.2 The modular group

**Definition 1.2.1.** The **modular group** is the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{Z}, \det(A) = 1 \right\}.$$

The **projective modular group** is  $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/(\pm 1)$ .

**Theorem 1.2.2.** (a) The group  $\mathrm{SL}_2(\mathbb{Z})$  is generated by  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

(b) Every orbit of  $\mathrm{SL}_2(\mathbb{Z})$  acting on  $\mathcal{H}$  contains a point of the set  $D$  defined by

$$D = \left\{ z \in \mathcal{H}: -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1 \right\}.$$

(c) If  $z \in D$  and  $gz \in D$  for some  $g \in \mathrm{SL}_2(\mathbb{Z})$ , then either  $g = \pm 1$  and  $gz = z$  or  $z$  lies on the boundary of  $D$ .

(d) The stabilizer of  $z \in \mathcal{H}$  in  $\mathrm{PSL}_2(\mathbb{Z})$  is trivial unless  $z$  is in the orbit of  $i$  or in the orbit of  $\rho = e^{2\pi i/3}$ .

*Proof.* We will prove all of these statements in 4 steps using a very elegant argument of Serre. Let  $G = \mathrm{SL}_2(\mathbb{Z})$  and  $G' = \langle S, T \rangle \leq G$ .

**Step 1.** Every  $G'$  orbit in  $\mathcal{H}$  contains a point of  $D$ .

*Proof of Step 1.* Let  $z \in \mathcal{H}$ . Since  $|cz+d| \geq |c \Im(z)|$  and  $|cz+d| \geq |c \Re(z)+d|$  there exist only finitely many  $(c, d) \in \mathbb{Z}^2$  such that  $|cz+d| < 1$ . Recall  $\Im\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = |cz+d|^{-2} \Im(z)$ . This implies there are only finitely many  $g \in G'$  such that  $\Im(gz) > \Im(z)$ . So the  $G'$  orbit of  $z$  contains a point of maximal imaginary part. Let this point be  $z$ .

We can assume  $\Re(z) \in [-\frac{1}{2}, \frac{1}{2}]$  since  $Tz = z + 1$ . Moreover  $\Im(Sz) = |z|^{-2} \Im(z)$ . But  $z$  is a point of maximal imaginary part in the orbit of  $G'$ , so we get  $|z|^{-2} \Im(z) \leq \Im(z)$  implying  $|z| \geq 1$ . Thus  $z \in D$ . Clearly this proves part (b) of the theorem.  $\square$

**Step 2.** If  $z \in D$  and  $gz \in D$ , where  $g \in G$ , then one of the following holds:

1.  $g = \pm \text{Id}$
2.  $g = \pm S$  and  $|z| = 1$
3.  $g = \pm T$  and  $\Re(z) = -\frac{1}{2}$ , or  $g = \pm T^{-1}$  and  $\Re(z) = \frac{1}{2}$
4.  $g = \pm ST = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$  or  $g = \pm T^{-1}S = \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$  or  $g = \pm ST^{-1}S = \pm \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$  and  $z = \rho$
5.  $g = \pm TS = \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$  or  $g = \pm ST^{-1} = \pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  or  $g = \pm STS = \pm \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$  and  $z = \rho + 1$

*Proof of Step 2.* Let  $z \in D$  and  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$  such that  $z' = gz \in D$ . Being free to replace  $g$  by  $g^{-1}$  and  $z$  by  $z'$  we can assume that  $\Im(z') \geq \Im(z)$ . Again recalling  $\Im(gz) = |cz + d|^{-2} \Im(z)$  we gain  $|cz + d| \leq 1$ . Furthermore we have

$$|cz + d| \geq |c| \Im(z) \geq |c| \Im(\rho) = \frac{\sqrt{3}}{2} |c|.$$

Thus  $|c| \leq 2/\sqrt{3} < 2$ . As  $c \in \mathbb{Z}$  we get  $c = 0$  or  $c = \pm 1$ .

- Let  $c = 0$ . Since  $1 \geq |cz + d| = |d|$  we have  $d = 0$  or  $d = \pm 1$ . But  $c = d = 0$  is impossible. So  $d = \pm 1$  and hence  $a = \pm 1$ . Therefore  $g = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$  is the translation by  $b$ . But since

$$\Re(z), \Re(gz) \in \left[ -\frac{1}{2}, \frac{1}{2} \right],$$

this implies that  $b = 0$  or  $b = \pm 1$ . So either  $g = \pm \text{Id}$  (case 1) or  $g = \pm T$  and  $\Re(z) = -\frac{1}{2}$  or  $g = \pm T^{-1}$  and  $\Re(z) = \frac{1}{2}$ .

- Let  $c = 1$ . Assuming  $|d| \geq 2$  leads to the following contradiction:

$$1 \geq |cz + d| = |z + d| \geq |d| - \Re(z) \geq |d| - \frac{1}{2} \geq \frac{3}{2}$$

Thus we have  $d = 0$  or  $d = \pm 1$ .

Let  $d = 0$ . Then  $1 \geq |cz + d| = |z|$ . On the other hand  $|z| \geq 1$  as  $z \in D$  and therefore  $|z| = 1$  (cases 2, 4 or 5 – exercise sheet 1).

Let  $d = 1$ . Then  $1 \geq |z + 1|$ . This is only possible for  $z \in D$  if  $z = \rho$  (exercise). Since  $a - b = 1$ , we deduce that wither  $(a, b) = (1, 0)$  or  $(a, b) = (0, -1)$  (case 4).

Analogue  $d = -1$  implies  $z = \rho + 1$  (case 5).

- The case  $c = -1$  is analogous to the case  $c = 1$ .

Since there are no further cases this shows Step 2 (it remains to check the matrices in case 4 and 5 – see exercise sheet 1) and therefore part (c) of the theorem.  $\square$

**Step 3.** Let  $z \in D$  such that the stabilizer  $G_z$  of  $z$  is not  $\pm \text{Id}$ . Then  $z = i$ ,  $z = \rho$  or  $z = \rho + 1$ .

*Proof of Step 3.* This follows directly from Step 2 by checking  $gz = z$  for all possible  $g$ 's. Step 3 proves part (d) of the theorem.  $\square$

**Step 4.** It remains to show that  $SL_2(\mathbb{Z})$  is generated by  $S$  and  $T$ .

*Proof of Step 4.* Let  $g \in G$  and let  $z$  be an arbitrary point of the interior of  $D$ . Then  $gz \in \mathcal{H}$  and by Step 1 exists  $g' \in G'$  such that  $g'(gz) \in D$ . Moreover Step 2 implies that either  $g'g \in \{\pm \text{Id}\}$  or  $z$  is on the boundary of  $D$  which is by assumption not the case. Thus either  $g'g = \text{Id}$  or  $g'g = -\text{Id}$ . Since  $S^2 = -\text{Id} \in G'$ , we deduce that  $g \in G'$ , so  $SL_2(\mathbb{Z})$  is generated by  $S$  and  $T$ . This proves part (a) of the theorem.  $\square$

Therefore the theorem is proved.  $\square$

**Remark 1.2.3.** We have seen in the proof of Theorem 1.2.2 that  $SL_2(\mathbb{Z})$  is generated by the elements  $S$  and  $T$ . These satisfy the relations

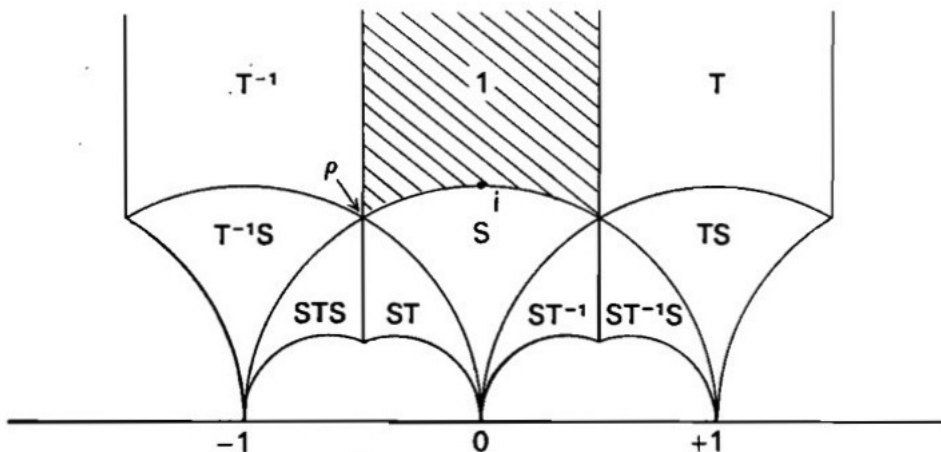
$$S^4 = \text{Id} \quad (ST)^3 = S^2,$$

and one can show that these generate all the relations, i.e. that

$$\langle S, T \mid S^4, S^{-2}(ST)^3 \rangle$$

is a presentation of the group  $SL_2(\mathbb{Z})$ .

**Remark 1.2.4.** The set  $D$  is called the **fundamental domain**. The figure below represents  $D$  itself and the transforms of  $D$  by some group elements of  $SL_2(\mathbb{Z})$ . Part (c) of the theorem shows that two sets  $gD$  and  $g'D$  where  $g, g' \in SL_2(\mathbb{Z})$  are either equal (if  $g' = \pm g$ ) or only intersect along their edges. Furthermore part (a) implies that  $\mathcal{H}$  is covered by the sets  $\{gD : g \in SL_2(\mathbb{Z})\}$ : they form a **tessellation** of  $\mathcal{H}$ .





### 1.3 Modular forms and modular functions

**Definition 1.3.1.** A weakly modular function of weight  $k$  and level 1 is a meromorphic function  $\mathcal{H} \rightarrow \mathbb{C}$  such that  $f|_k \alpha = f$  for all  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ , or equivalent

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all  $z \in \mathcal{H}$  and for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ .

**Note 1.3.2.** Since  $\mathrm{SL}_2(\mathbb{Z})$  is generated by the matrices  $S$  and  $T$ , it is sufficient to check invariance under these two matrices, i.e. that

$$f(z+1) = f(z) \quad \text{and} \quad f(-1/z) = z^k f(z)$$

for all  $z \in \mathcal{H}$ .

**Lemma 1.3.3.** *There are no nonzero weakly modular functions of odd weight.*

*Proof.* Let  $k$  be odd and let  $f$  be a weakly modular function of weight  $k$ . As shown in (2) we have  $f(z) = f(z+1)$  for all  $z \in \mathcal{H}$ . Moreover we get  $f(z) = -f(z+1)$  for all  $z \in \mathcal{H}$ , since  $f|_k \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = -f(\cdot + 1)$ . So  $f(z) = -f(z)$  and thus  $f(z) = 0$  for all  $z \in \mathcal{H}$ .  $\square$

Define the function

$$\begin{aligned} q : \mathcal{H} &\rightarrow \mathbb{C}, \\ z &\mapsto \exp(2\pi iz). \end{aligned}$$

**Note 1.3.4.** Now let  $f$  be weakly periodic of weight  $k$ . Then  $f$  is periodic with period 1, so it can be written in the form

$$f(z) = \tilde{f}(\exp(2\pi iz)),$$

where  $\tilde{f}$  is a meromorphic function on the punctured unit disk

$$\mathbb{D}^* = \{q \in \mathbb{C} : 0 < |q| < 1\}.$$

**Note 1.3.5.** The function  $\tilde{f}$  is defined by

$$\tilde{f}(q) = f\left(\frac{\log q}{2\pi i}\right).$$

Observe that the logarithm is multi-valued, but choosing a different value of the logarithm is the same as adding an integer to  $\frac{\log q}{2\pi i}$ . The periodicity of  $f$  hence implies that  $\tilde{f}(q)$  does not depend on the chosen value of the logarithm.

**Note 1.3.6.** Any weakly modular function can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

for some  $a_n \in \mathbb{C}$  where  $q = e^{2\pi iz}$ ; we call this the  $q$ -*expansion of  $f$* . This is just the Laurent series of  $\tilde{f}$  around  $q = 0$ , which converges for  $0 < |q| < \varepsilon$  for  $\varepsilon$  sufficiently small ( $\Leftrightarrow \Im(z) \gg 0$ )

**Definition 1.3.7.**

- We say that  $f$  is meromorphic at  $\infty$  if  $a_n = 0$  for  $n < -N$  and some  $N \in \mathbb{N}$ .
- We say that  $f$  is holomorphic at  $\infty$  if  $a_n = 0$  for  $n < 0$ . In this case, we define the value of  $f$  at  $\infty$  to be  $f(\infty) = \tilde{f}(0) = a_0$ .

**Definition 1.3.8.** Let  $f$  be a weakly modular function of weight  $k$  and level 1.

1. If  $f$  is meromorphic on  $\mathcal{H} \cup \{\infty\}$  we say  $f$  is a **modular function** (of weight  $k$  and level 1).
2. If  $f$  is holomorphic on  $\mathcal{H} \cup \{\infty\}$  we say  $f$  is a **modular form** (of weight  $k$  and level 1).
3. If  $f$  is holomorphic on  $\mathcal{H} \cup \{\infty\}$  and  $f(\infty) = 0$  we say  $f$  is a **cuspidal modular form** or **cusp form**.

**Note 1.3.9.** If  $f$  and  $g$  are modular forms (resp. modular functions) of level 1 and weights  $k$  and  $\ell$ , then the product  $fg$  is a modular form (resp. modular function) of weight  $k + \ell$ .

## 1.4 Eisenstein series

**Definition 1.4.1.** Let  $k \geq 4$  even. Define the **Eisenstein series of weight  $k$**  to be the function  $G_k: \mathcal{H} \rightarrow \mathbb{C}$  given by

$$G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(mz + n)^k}. \quad (1.1)$$

Recall the following result from complex analysis:

**Proposition 1.4.2.** *Let  $U$  be an open subset of  $\mathbb{C}$ , and let  $(f_n)_n \geq 0$  be a sequence of holomorphic functions on  $U$  that converges uniformly on compact subsets of  $U$ . Then the limit function  $U \rightarrow \mathbb{C}$  is holomorphic.*

**Lemma 1.4.3.** *The series defining  $G_k(z)$  converges absolutely and uniformly on subsets of  $\mathcal{H}$  of the form*

$$R_{r,s} = \{x + iy : |x| \leq r, y \geq s\}.$$

*It hence converges to a holomorphic function on  $\mathcal{H}$ .*

*Proof.* Let  $z = x + iy \in R_{r,s}$ . We have

$$|mz + n|^2 = (mx + n)^2 + m^2y^2 \geq (mx + n)^2 + m^2s^2.$$

For fixed  $m$  and  $n$ , we distinguish the cases  $|n| \leq 2r|m|$  and  $|n| \geq 2r|m|$ . In the first case, we have

$$|mz + n|^2 \geq m^2s^2 \geq \frac{s^2}{2}m^2 + \frac{s^2}{2(2r)^2}n^2 \geq \min\left\{\frac{s^2}{2}, \frac{s^2}{8r^2}\right\} \cdot (m^2 + n^2).$$

In the second case, the triangle inequality implies

$$|mz + n|^2 \geq (|mx| - |n|)^2 + m^2s^2 \geq \left(\frac{|n|}{2}\right)^2 + m^2s^2 \geq \min\left\{\frac{1}{4}, s^2\right\} \cdot (m^2 + n^2).$$

Combining both cases and putting

$$c = \min\left\{\frac{s^2}{2}, \frac{s^2}{8r^2}, \frac{1}{4}, s^2\right\},$$

we get the inequality

$$|mz + n| \geq c^{1/2}(m^2 + n^2)^{1/2} \quad \text{for all } m, n \in \mathbb{Z}, z \in R_{r,s}.$$

Hence for all  $z \in R_{r,s}$ , we have

$$G_k(z) \leq \frac{1}{c^{k/2}} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^{k/2}}.$$

We rearrange the sum by grouping together, for each fixed  $j = 1, 2, 3, \dots$ , all pairs  $(m, n)$  with  $\max\{|m|, |n|\} = j$ . We note that for each  $j$  there are  $8j$  such pairs  $(m, n)$ , each of which satisfies

$$j^2 \leq m^2 + n^2.$$

Hence

$$|G_k(z)| \leq \frac{1}{c^{k/2}} \sum_{j=1}^{\infty} \frac{8j}{j^k} = \frac{8}{c^{k/2}} \sum_{j=1}^{\infty} \frac{1}{j^{k-1}},$$

which is finite and independent of  $z \in R_{r,s}$ , so  $G_k(z)$  converges absolutely and uniformly on  $R_{r,s}$ . Since every compact subset of  $\mathcal{H}$  is contained in some  $R_{r,s}$ , this finishes the proof by Proposition 1.4.2.  $\square$

**Remark 1.4.4.** This proof clearly fails for  $k = 2$ . One can show that for  $k = 2$ , the series (1.1) is conditionally but not absolutely convergent. We will come back to this issue later in the course.

**Proposition 1.4.5.** *For every even integer  $k \geq 4$ , the function  $G_k$  is a modular form of weight  $k$  and level 1. The  $q$ -expansion of  $G_k$  is given by*

$$G_k(z) = 2 \zeta(k) + \frac{2 \cdot (2\pi i)^k}{(k-1)!} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where  $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$  (the Riemann zeta function) and  $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ .

*Proof.* One easily checks that  $G_k(z+1) = G_k(z)$ . Moreover, we have

$$\begin{aligned} G_k\left(-\frac{1}{z}\right) &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m(-\frac{1}{z}) + n)^k} \\ &= z^k \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(-m + nz)^k} \\ &= z^k G_k(z). \end{aligned}$$

Hence  $G_k|_k S = G_k$  and  $G_k|_k T = G_k$ , so  $G_k|_k \alpha = G_k$  for all  $\alpha \in \text{SL}_2(\mathbb{Z})$  by Theorem 1.2.2 (a). Thus  $G_k$  is a weakly modular function of weight  $k$  and level 1.

It remains to show that  $G_k$  is holomorphic at  $\infty$ . Therefore we will determine the  $q$ -expansion of  $G_k$ . Consider the formula  $\sum_{n \in \mathbb{Z}} \frac{1}{z+n} = \pi \cdot \cot(\pi z)$ . Thus we obtain

$$\sum_{n \in \mathbb{Z}} \frac{1}{z+n} = \pi \cdot \cot(\pi z) = i\pi \left( \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} \right) = i\pi \left( 1 + \frac{2}{q-1} \right) = i\pi - 2\pi i \sum_{n=0}^{\infty} q^n,$$

where  $q = e^{2\pi iz}$ . Differentiating  $(k-1)$  times with respect to  $z$ , and using that  $\frac{\partial}{\partial z} = 2\pi i q \frac{\partial}{\partial q}$ , leads to

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{-(k-1)!}{(z+n)^k} &= \frac{\partial^{k-1}}{\partial z^{k-1}} \left( i\pi - 2\pi i \sum_{n=0}^{\infty} q^n \right) \\ &= -2\pi i \sum_{n=1}^{\infty} (2\pi i n)^{k-1} q^n \\ &= -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n \end{aligned}$$

(We are using here that  $k$  is even; for  $k$  odd we get an additional  $-\text{sign}$ .)

Hence we get

$$t_k(z) := \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}.$$

Now we can split up the original sum of the function  $G_k$  into two parts, one where  $m = 0$  and one where  $m \neq 0$ . Afterwards we will simplify both parts using symmetry (remember again that  $k$  is even) of the sums and the above formula:

$$\begin{aligned}
G_k(z) &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^k} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^k} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^k} \\
&= 2\zeta(k) + 2 \sum_{m=1}^{\infty} t_k(mz) \\
&= 2\zeta(k) + 2 \sum_{m=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z} \\
&= 2\zeta(k) + \frac{2 \cdot (2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm}
\end{aligned}$$

From there we obtain the proposed  $q$ -expansion by resorting the last sum:

$$G_k(z) = 2\zeta(k) + \frac{2 \cdot (2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} \underbrace{\sum_{d|l} d^{k-1}}_{\sigma_{k-1}(l)} q^l$$

And since  $G_k$  has a  $q$ -expansion without any negative powers of  $q$ ,  $G_k$  is holomorphic at  $\infty$ . Thus  $G_k$  is indeed a modular form.  $\square$

**Definition 1.4.6.** The Bernoulli numbers are the rational numbers  $B_k$ , for  $k \geq 0$ , defined by the equation

$$\frac{t}{\exp(t) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \in \mathbb{Q}[[t]].$$

**Remark 1.4.7.** The Bernoulli numbers are of great importance in mathematics. Barry Mazur once said: “When a Bernoulli number sneezes, the tremors can be felt in all of mathematics.”

**Lemma 1.4.8.** We have

$$B_k \neq 0 \quad \Leftrightarrow \quad k = 1 \text{ or } k \text{ is even.}$$

*Proof.* Exercise sheet 2.  $\square$

**Example 1.4.9.** The first few non-zero Bernoulli numbers

$$\begin{aligned}
B_0 = 0, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{3}, \quad B_6 = \frac{1}{42}, \\
B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.
\end{aligned}$$

**Lemma 1.4.10.** *If  $k \geq 2$  is an even integer, then*

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}.$$

*Proof.* Exercise sheet 2. □

**Definition 1.4.11.** Let  $k \geq 4$  be even. The normalised **Eisenstein series** of weight  $k$  is given by

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

## 1.5 The valence formula

**Definition 1.5.1.** Let  $f \neq 0$  be a meromorphic function  $\mathcal{H} \rightarrow \mathbb{C}$  and let  $P \in \mathcal{H}$ . The unique integer  $n$  such that  $(z-P)^{-n} f(z)$  is holomorphic and non-vanishing at  $P$  is called the **order of  $f$  at  $P$**  and denoted by  $v_P(f)$ . We say  $f$  has a **zero of order  $n$  at  $P$**  if  $n$  is positive, and  $f$  has a **pole of order  $n$  at  $P$**  if  $n$  is negative.

**Definition 1.5.2.** Consider the Laurent expansion of  $f$  around  $P$

$$f(z) = \sum_{n \geq n_0} c_n (z-P)^n.$$

Then the **residue of  $f$  at  $P$**  is  $\text{Res}_P(f) = c_{-1} \in \mathbb{C}$ .

**Lemma 1.5.3.** *If  $f$  is meromorphic around a point  $P$ , then*

$$\text{Res}_P(f/f') = v_P(f).$$

*Proof.* Exercise. □

We recall without proof the following results from complex analysis:

**Theorem 1.5.4.** *(Cauchy's integral formula) Let  $g$  be a holomorphic function on an open subset  $U \subseteq \mathbb{C}$  and let  $C$  be a contour in  $U$ . Then for each  $P \in U$ , we have*

$$\int_C \frac{g(z)}{z-P} dz = 2\pi i \cdot g(P).$$

**Corollary 1.5.5.** *Let  $C(P, r, \alpha)$  be an arc of a circle of radius  $r$  and angle  $\alpha$  around a point  $P$ . If  $g$  is holomorphic at  $P$ , then*

$$\lim_{r \rightarrow 0} \int_{C(P, r, \alpha)} \frac{g(z)}{z-P} dz = \alpha i \cdot g(P).$$

*(Here, we integrate counterclockwise.)*

The following result relates the contour integral of the logarithmic derivative of  $f$  to the orders of  $f$  at the interior points:

**Theorem 1.5.6.** (*Argument principle*) *Let  $f$  be a meromorphic function on an open subset  $U \subseteq \mathbb{C}$ , and let  $C$  be a contour in  $U$  not passing through any zeros or poles of  $f$ . Then*

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \text{int}(C)} v_P(f).$$

**Note 1.5.7.** By Lemma 1.5.3, we have

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \text{int}(C)} \text{Res}_P(f'/f). \quad (1.2)$$

**Corollary 1.5.8.** *Let  $C(P, r, \alpha)$  be an arc of a circle of radius  $r$  and angle  $\alpha$  around a point  $P$ . If  $f$  is meromorphic at  $P$ , then*

$$\lim_{r \rightarrow 0} \int_{C(P, r, \alpha)} \frac{f'(z)}{f(z)} dz = \alpha i \cdot v_P(f).$$

Now assume that  $f$  is a weakly modular function (of weight  $k$  and level 1).

**Remark 1.5.9.** Since  $f|_k \alpha = f$  for all  $\alpha \in \text{SL}_2(\mathbb{Z})$ , we have  $v_{\alpha P}(f) = v_P(f)$ . Hence  $v_P(f)$  is well-defined for  $P$  being a  $\text{SL}_2(\mathbb{Z})$  orbit in  $\mathcal{H}$ .

Moreover, if  $f$  is meromorphic at  $\infty$ , we can define the order of  $f$  at  $\infty$  by

$$v_\infty(f) := v_0(\tilde{f}).$$

The following theorem is fundamental for studying the spaces of modular forms:

**Theorem 1.5.10.** (*The valence formula*) *Let  $f \neq 0$  be a modular function of weight  $k$  and level 1. Then  $f$  has finitely many  $\text{SL}_2(\mathbb{Z})$ -orbits of zeros and poles in  $\mathcal{H}$ , and*

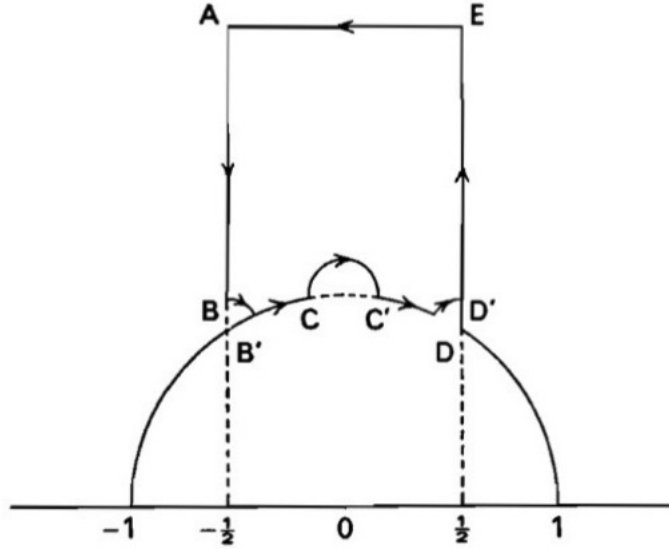
$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{P \in W} v_P(f) = \frac{k}{12}, \quad (1.3)$$

where  $\rho = e^{2\pi i/3}$  and  $W$  is the set of all  $\text{SL}_2(\mathbb{Z})$ -orbits in  $\mathcal{H}$  except the orbits of  $i$  and  $\rho$ .

*Proof.* Recall the fundamental domain from 1.2.2 and let  $\mathcal{C}$  be the contour as shown in the figure below. Here  $\Im(A) = \Im(E) = R$  (we will later let  $R \rightarrow +\infty$ ) and the three small circles have radius  $r$ . We assume that  $R$  is sufficiently large and  $r$  sufficiently small that the interior of  $\mathcal{C}$  contains all the zeros and poles of  $f$  except those at  $i$ ,  $\rho$ ,  $\rho + 1$  and  $\infty$ .

*Simplifying assumption:* We assume for simplicity  $f$  has no zeros or poles on the boundary of the fundamental domain, except possibly at  $i$  and  $\rho$ . (In the case where it does contain zeros or poles of  $f$ , the contour has to be modified using additional small arcs going around these zeros or poles in the counterclockwise direction.)

We will now calculate  $\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz$  in two different ways and compose the results afterwards.



(1) Computing the integral using Theorem 1.5.6, we get

$$\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \text{interior}(\mathcal{C})} v_P(f) = 2\pi i \sum_{P \in W} v_P(f),$$

where  $W$  is the set described in the stated theorem. The last equality is satisfied by the simplifying assumption, so the interior of the fundamental domain contains exactly one representative of every pole or zero  $\text{SL}_2(\mathbb{Z})$ -orbit of  $\mathcal{H}$ .

(2) Secondly, we estimate the integral by splitting up the contour in 8 parts. Let  $\mathcal{C}_1$  be the part from  $E$  to  $A$ ,  $\mathcal{C}_2$  be the part from  $A$  to  $B$ , and so on, such that in the end  $\mathcal{C}_8$  is the part from  $D'$  to  $E$ .

(i) Note that since  $f$  is a modular function, we have  $f(z) = f(z+1)$ . Hence also  $f'(z) = f'(z+1)$ , and we have

$$\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)} dz = \int_{\mathcal{C}_2} \frac{f'(z+1)}{f(z+1)} dz = - \int_{\mathcal{C}_8} \frac{f'(z)}{f(z)} dz,$$

so

$$\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)} dz + \int_{\mathcal{C}_8} \frac{f'(z)}{f(z)} dz = 0.$$

(ii) Now we consider  $\mathcal{C}_1$  and change the variable by  $q(z) = e^{2\pi iz}$ . This maps  $\mathcal{C}_1$  to a clockwise oriented circle around the origin with radius  $e^{-2\pi R}$ . Furthermore we have  $f(z) = \tilde{f}(q(z))$ , thus  $f'(z) = \tilde{f}'(q(z)) q'(z)$  and since  $f$  is a modular



function,  $\tilde{f}$  is meromorphic at 0. Therefore

$$\begin{aligned}
\int_{\mathcal{C}_1} \frac{f'(z)}{f(z)} dz &= \int_{\mathcal{C}_1} \frac{\tilde{f}'(q(z))q'(z)}{\tilde{f}(q(z))} dz \\
&= \int_{q(\mathcal{C}_1)} \frac{\tilde{f}'(q)}{\tilde{f}(q)} dq \\
&= -2\pi i \operatorname{Res}_0 \left( \frac{\tilde{f}'}{\tilde{f}} \right) \\
&= -2\pi i v_0(\tilde{f}) \\
&= -2\pi i v_\infty(f).
\end{aligned}$$

(iii)  $\mathcal{C}_5$  is half of a circle around  $i$ . We deduce from Corollary 1.5.8 that

$$\lim_{r \rightarrow 0} \int_{\mathcal{C}_5} \frac{f'(z)}{f(z)} dz = -\frac{1}{2} 2\pi i v_i(f).$$

Similarly we get

$$\begin{aligned}
\lim_{r \rightarrow 0} \int_{\mathcal{C}_3} \frac{f'(z)}{f(z)} dz &= -\frac{1}{6} 2\pi i v_\rho(f) \\
\lim_{r \rightarrow 0} \int_{\mathcal{C}_7} \frac{f'(z)}{f(z)} dz &= -\frac{1}{6} 2\pi i v_{\rho+1}(f) = -\frac{1}{6} 2\pi i v_\rho(f).
\end{aligned}$$

(iv) So it remains to study  $\mathcal{C}_4$  and  $\mathcal{C}_6$ . Therefore consider  $u(z) = -\frac{1}{z}$ . This maps  $\mathcal{C}_6$  to  $-\mathcal{C}_4$  and we have  $f(z) = z^{-k} f(u(z))$ , hence

$$f'(z) = -kz^{-k-1} f(u(z)) + z^{-k} f'(u(z))u'(z).$$

So

$$\begin{aligned}
\int_{\mathcal{C}_4} \frac{f'(z)}{f(z)} dz &= \int_{\mathcal{C}_4} \frac{-k}{z} dz + \int_{\mathcal{C}_4} \frac{f'(u(z))u'(z)}{f(u(z))} dz \\
&= \frac{2\pi i k}{12} + \int_{u(\mathcal{C}_4)} \frac{f'(u)}{f(u)} du \\
&= \frac{2\pi i k}{12} - \int_{\mathcal{C}_6} \frac{f'(u)}{f(u)} du
\end{aligned}$$

and thus

$$\int_{\mathcal{C}_4} \frac{f'(z)}{f(z)} dz + \int_{\mathcal{C}_6} \frac{f'(z)}{f(z)} dz = 2\pi i \frac{k}{12}.$$

Composing (i) to (iv) yields

$$\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = 2\pi i \left( \frac{k}{12} - \frac{1}{3} v_\rho(f) - \frac{1}{2} v_i(f) - v_\infty(f) \right).$$

Combining this with the result in (1) gives us exactly the proposed formula.  $\square$