Modular Forms

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0 Prologue

Example 0.0.1. Let $z \in \mathbb{C}$, $\Im(z) > 0$. Let $q = e^{2\pi i z}$ and define **Ramanujan's tau function**

$$\Delta(z) = q \cdot \prod_{n \in \mathbb{N}} (1 - q^n)^{24}.$$

This is one of the simplest examples of a modular form. Note that we can "multiply out" the product above which leads us to

$$\Delta(z) = \sum_{n \in \mathbb{N}} \tau(n) q^n$$

for some integers $\tau(n)$.

Facts 0.0.2.

(1) Known to Weierstrass, 1850:

$$\Delta(z) = z^{-12} \cdot \Delta\left(-\frac{1}{z}\right)$$

(2) Ramanujan proved in 1916 that the integers $\tau(n)$ satisfy the equation

$$\tau(n) = \sum_{d|n} d^{11} \mod 691.$$

(3) Ramanujan also conjectured $\tau(nm) = \tau(n)\tau(m)$ for n, m coprime. This was proved by Mordell in 1917.

(4) In 1972 Swinnerton-Dyer proved $\tau(n)$ satisfies congruences like (2) modulo 2, 3, 5, 7, 23 and 691 but no other primes.

(5) Ramanujan conjectured in 1916 for p prime holds $|\tau(p)| < 2\, p^{11/2}$. This was proved in 1974 by Deligne.

(6) The quantity

$$\frac{\tau(p)}{2p^{11/2}} \in [-1, 1]$$

is distributed in the interval $[-1, 1]$ with density function proportional to $\sqrt{1 - x^2}$. This was conjectured by Sato and Tate (1960s) and proved by Barnet-Lamb, Geraghty, Harris and Taylor in 2009 using Bau Chau Ngo's *Fundamental Lemma* which got Ngo the 2010 Fields Medal.

Example 0.0.3. We now consider another modular form

$$\begin{aligned} f(z) &= q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 + \dots \\ &= \sum_{n=1}^{\infty} a(n)q^n \quad ' \quad \text{with } a(n) \in \mathbb{N} \end{aligned}$$

We will later prove the following results:

Theorem.

1. *We have $a(mn) = a(m)(n)$ for all $m, n \geq 1$ with $(m, n) = 1$.*
2. *We have $|a(p)| \leq 2\sqrt{p}$ for all primes p.*

It turns out that this modular form is closely related to the elliptic curve

$$E : Y^2 + Y = X^3 - X^2 - 10X - 20.$$

For p prime, denote by $N(p)$ the number of points on the elliptic curve in $\mathbb{F}_p$. It is easy to see heuristically tat $N(p) \simeq p$.

Theorem. *(Hasse) We have*

$$|p - N(p)| \leq 2\sqrt{p}.$$

The theory of modular forms allows one to prove that the elliptic curve E and the modular form f 'correspond' to each other in the following sense:

Theorem. *For all primes p, we have*

$$a(p) = p - N(p).$$

In particular, using the properties of the modular form f, we can easily calculate the quantity $N(p)$ for all p, so f 'knows' about the behaviour of the elliptic curve over $\mathbb{F}_p$. We say that the elliptic curve E is **modular**. It is generally not too difficult to attach an elliptic curve to a modular form (this is called "Eichler–Shimura"); however, it is very difficult indeed to reverse this process, and this is the basis of Andrew Wiles' work on Fermat's Last Theorem. The proof of this result was later completed by Breuil–Conrad–Diamond–Taylor. I will talk a bit more about this when we discuss L-functions of modular forms.

1 The modular group

1.1 The upper half-plane

Definition 1.1.1. Let $\mathcal{H} = \{z \in \mathbb{C} \colon \Im(z) > 0\}$ the **upper half-plane**.

Proposition 1.1.2. *The **special linear group** $\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \colon \det(A) = 1\}$ acts on $\mathcal{H}$ via*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}.$$

Proof. For $z \in \mathcal{H}$ is $\Im(z) > 0$ and either c or d is nonzero, so $cz + d \neq 0$. Moreover

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{1}{|cz+d|^2}\,\Im\left((az+b)(c\overline{z}+d)\right).$$

Say $z = x + iy$ for $x, y \in \mathbb{R}$.

$$\begin{aligned}\Im\left(\frac{az+b}{cz+d}\right) &= \frac{1}{|cz+d|^2}\,\Im\Big(\underbrace{(ax+b)(cx+d)+acy^2}_{\in\mathbb{R}} + i\underbrace{(ad-bc)}_{=1}y\Big)\\ &= \frac{1}{|cz+d|^2}\,\Im(z) \;>\; 0\end{aligned}$$

Therefore $\frac{az+b}{cz+d} \in \mathcal{H}$ for any $z \in \mathcal{H}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

Also it is easy to check that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)z = z$ and $A(Bz) = (AB)z$ for any $z \in \mathcal{H}$ and for any $A, B \in \mathrm{SL}_2(\mathbb{R})$. Thus $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$. □

Note 1.1.3. The matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ acts trivially on $\mathcal{H}$, so the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ factors through the quotient $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/(\pm 1)$, the **projective special linear group**.

Definition 1.1.4. The *automorphy factor* is the function

$$\begin{aligned} j \,:\, \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H} &\to \mathbb{C},\\ (g, z) &\mapsto cz + d \qquad\qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\end{aligned}$$

Proposition 1.1.5. *For any $k \in \mathbb{Z}$, we can define a right action of $\mathrm{SL}_2(\mathbb{R})$ on the set of holomorphic functions $\mathcal{H} \to \mathbb{C}$ given by*

$$(f|_k g)(z) := j(g,z)^{-k}\; f(gz)$$

*where $f \colon \mathcal{H} \to \mathbb{C}$ holomorphic, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. We will call this the **weight k action**.*

Proof. Firstly we need to show that $f|_k g$ is a well-defined holomorphic function $\mathcal{H} \to \mathbb{C}$. But this is obvious since $cz + d \neq 0$ and $gz \in \mathcal{H}$ for all $z \in \mathcal{H}$. Clearly also the equation $f|_k 1 = f$ holds. Therefore it remains to show $(f|_k g)|_k h = f|_k(gh)$ for arbitrary $g, h \in \mathrm{SL}_2(\mathbb{R})$. The left hand side of the equation can be rewritten as

$$\begin{aligned}(f|_k g)|_k h &= j(h,z)^{-k}\left((f|_k g)(hz)\right)\\ &= j(h,z)^{-k} j(g,hz)^{-k} f(g(hz))\end{aligned}$$

and the right hand side results in

$$f|_k(gh) = j(gh,z)^{-k} f((gh)z).$$

We already know $(gh)z = g(hz)$. So it remains to show $j(gh,z) = j(h,z)j(g,hz)$. This is the so called **cocycle relation** and can be checked easily. □

1.2 The modular group

Definition 1.2.1. The **modular group** is the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a,b,c,d \in \mathbb{Z},\ \det(A) = 1 \right\}.$$

The **projective modular group** is $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/(\pm 1)$.

Theorem 1.2.2. *(a) The group* $\mathrm{SL}_2(\mathbb{Z})$ *is generated by* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(b) Every orbit of $\mathrm{SL}_2(\mathbb{Z})$ *acting on* $\mathcal{H}$ *contains a point of the set* D *defined by*

$$D = \left\{ z \in \mathcal{H} \colon\ -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1 \right\}.$$

(c) If $z \in D$ *and* $gz \in D$ *for some* $g \in \mathrm{SL}_2(\mathbb{Z})$*, then either* $g = \pm 1$ *and* $gz = z$ *or* z *lies on the boundary of* D.

(d) The stabilizer of $z \in \mathcal{H}$ *in* $\mathrm{PSL}_2(\mathbb{Z})$ *is trivial unless* z *is in the orbit of* i *or in the orbit of* $\rho = e^{2\pi i/3}$.

Proof. We will prove all of these statements in 4 steps using a very elegant argument of Serre. Let $G = \mathrm{SL}_2(\mathbb{Z})$ and $G' = \langle S, T \rangle \leq G$.

Step 1. Every G' orbit in $\mathcal{H}$ contains a point of D.

Proof of Step 1. Let $z \in \mathcal{H}$. Since $|cz+d| \geq |c\,\Im(z)|$ and $|cz+d| \geq |c\,\Re(z)+d|$ there exist only finitely many $(c,d) \in \mathbb{Z}^2$ such that $|cz+d| < 1$. Recall $\Im(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)z) = |cz+d|^{-2}\,\Im(z)$. This implies there are only finitely many $g \in G'$ such that $\Im(gz) > \Im(z)$. So the G' orbit of z contains a point of maximal imaginary part. Let this point be z.

We can assume $\Re(z) \in [-\frac{1}{2}, \frac{1}{2}]$ since $Tz = z + 1$. Moreover $\Im(Sz) = |z|^{-2}\,\Im(z)$. But z is a point of maximal imaginary part in the orbit of G', so we get $|z|^{-2}\,\Im(z) \leq \Im(z)$ implying $|z| \geq 1$. Thus $z \in D$. Clearly this proves part (b) of the theorem. □

Step 2. If $z \in D$ and $gz \in D$, where $g \in G$, then one of the following holds:

1. $g = \pm \mathrm{Id}$
2. $g = \pm S$ and $|z| = 1$
3. $g = \pm T$ and $\Re(z) = -\frac{1}{2}$, or $g = \pm T^{-1}$ and $\Re(z) = \frac{1}{2}$
4. $g = \pm ST = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ or $g = \pm T^{-1}S = \pm \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $g = \pm ST^{-1}S = \pm \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ and $z = \rho$
5. $g = \pm TS = \pm \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $g = \pm ST^{-1} = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ or $g = \pm STS = \pm \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and $z = \rho + 1$

Proof of Step 2. Let $z \in D$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ such that $z' = gz \in D$. Being free to replace g by g^{-1} and z by z' we can assume that $\Im(z') \geq \Im(z)$. Again recalling $\Im(gz) = |cz+d|^{-2}\, \Im(z)$ we gain $|cz + d| \leq 1$. Furthermore we have

$$|cz + d| \geq |c|\ \Im(z) \geq |c|\ \Im(\rho) = \frac{\sqrt{3}}{2}|c|.$$

Thus $|c| \leq 2/\sqrt{3} < 2$. As $c \in \mathbb{Z}$ we get $c = 0$ or $c = \pm 1$.

- Let $c = 0$. Since $1 \geq |cz + d| = |d|$ we have $d = 0$ or $d = \pm 1$. But $c = d = 0$ is impossible. So $d = \pm 1$ and hence $a = \pm 1$. Therefore $g = \left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$ is the translation by b. But since
$$\Re(z),\, \Re(gz) \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$
this implies that $b = 0$ or $b = \pm 1$. So either $g = \pm \mathrm{Id}$ (case 1) or $g = \pm T$ and $\Re(z) = -\frac{1}{2}$ or $g = \pm T^{-1}$ and $\Re(z) = \frac{1}{2}$.

- Let $c = 1$. Assuming $|d| \geq 2$ leads to the following contradiction:
$$1 \geq |cz + d| = |z + d| \geq |d| - \Re(z) \geq |d| - \frac{1}{2} \geq \frac{3}{2}$$
Thus we have $d = 0$ or $d = \pm 1$.

  Let $d = 0$. Then $1 \geq |cz + d| = |z|$. On the other hand $|z| \geq 1$ as $z \in D$ and therefore $|z| = 1$ (cases 2, 4 or 5 – exercise sheet 1).

  Let $d = 1$. Then $1 \geq |z + 1|$. This is only possible for $z \in D$ if $z = \rho$ (exercise). Since $a - b = 1$, we deduce that wither $(a, b) = (1, 0)$ or $(a, b) = (0, -1)$ (case 4).

  Analogue $d = -1$ implies $z = \rho + 1$ (case 5).

- The case $c = -1$ is analogous to the case $c = 1$.

Since there are no further cases this shows Step 2 (it remains to check the matrices in case 4 and 5 – see exercise sheet 1) and therefore part (c) of the theorem. □

Step 3. Let $z \in D$ such that the stabilizer G_z of z is not $\pm\mathrm{Id}$. Then $z = i$, $z = \rho$ or $z = \rho + 1$.

Proof of Step 3. This follows directly from Step 2 by checking $gz = z$ for all possible g's. Step 3 proves part (d) of the theorem. □

Step 4. It remains to show that $SL_2(\mathbb{Z})$ is generated by S and T.

Proof of Step 4. Let $g \in G$ and let z be an arbitrary point of the interior of D. Then $gz \in \mathcal{H}$ and by Step 1 exists $g' \in G'$ such that $g'(gz) \in D$. Moreover Step 2 implies that either $g'g \in \{\pm \mathrm{Id}\}$ or z is on the boundary of D which is by assumption not the case. Thus either $g'g = \mathrm{Id}$ or $g'g = -\mathrm{Id}$. Since $S^2 = -\mathrm{Id} \in G'$, we deduce that $g \in G'$, so $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T. This proves part (a) of the theorem. □

Therefore the theorem is proved. □

Remark 1.2.3. We have seen in the proof of Theorem 1.2.2 that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements S and T. These satisfy the relations

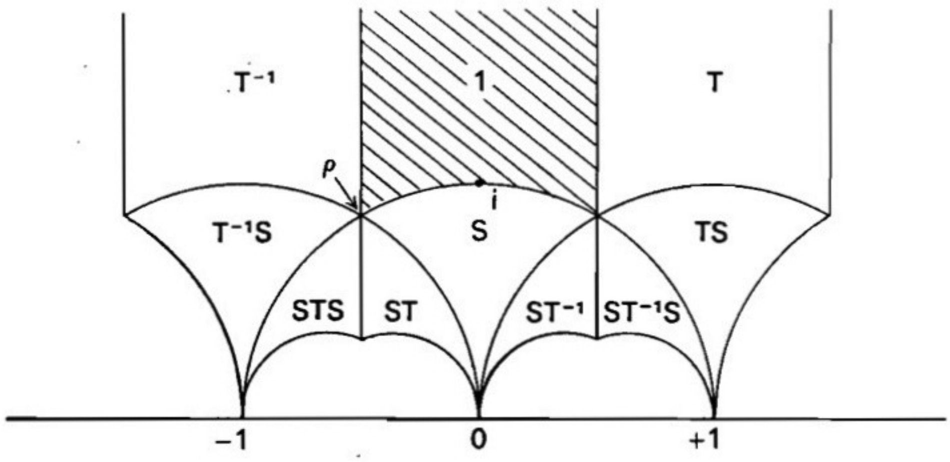
$$S^4 = \mathrm{Id} \quad (ST)^3 = S^2,$$

and one can show that these generate all the relations, i.e. that

$$\langle S, T \mid S^4,\, S^{-2}(ST)^3 \rangle$$

is a presentation of the group $\mathrm{SL}_2(\mathbb{Z})$.

Remark 1.2.4. The set D is called the **fundamental domain**. The figure below represents D itself and the transforms of D by some group elements of $\mathrm{SL}_2(\mathbb{Z})$. Part (c) of the theorem shows that two sets gD and $g'D$ where $g, g' \in \mathrm{SL}_2(\mathbb{Z})$ are either equal (if $g' = \pm g$) or only intersect along their edges. Furthermore part (a) implies that $\mathcal{H}$ is covered by the sets $\{gD\colon g \in \mathrm{SL}_2(\mathbb{Z})\}$: they form a **tesselation** of $\mathcal{H}$.



1.3 Modular forms and modular functions

Definition 1.3.1. A **weakly modular function of weight k and level 1** is a meromorphic function $\mathcal{H} \to \mathbb{C}$ such that $f|_k\alpha = f$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, or equivalent

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k\ f(z)$$

for all $z \in \mathcal{H}$ and for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Note 1.3.2. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices S and T, it is sufficient to check invariance under these two matrices, i.e. that

$$f(z+1) = f(z) \qquad \text{and} \qquad f(-1/z) = z^k f(z)$$

for all $z \in \mathcal{H}$.

Lemma 1.3.3. *There are no nonzero weakly modular functions of odd weight.*

Proof. Let k be odd and let f be a weakly modular function of weight k. As shown in (2) we have $f(z) = f(z+1)$ for all $z \in \mathcal{H}$. Moreover we get $f(z) = -f(z+1)$ for all $z \in \mathcal{H}$, since $f|_k\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right) = -f(\cdot + 1)$. So $f(z) = -f(z)$ and thus $f(z) = 0$ for all $z \in \mathcal{H}$. ☐

Define the function

$$\begin{aligned} q:\ \mathcal{H} &\to \mathbb{C}, \\ z &\mapsto \exp(2\pi i z). \end{aligned}$$

Note 1.3.4. Now let f be weakly periodic of weight k. Then f is periodic with period 1, so it can be written in the form

$$f(z) = \tilde{f}(\exp(2\pi i z)),$$

where $\tilde{f}$ is a meromorphic function on the punctured unit disk

$$\mathbb{D}^* = \{q \in \mathbb{C} :\ 0 < |q| < 1\}.$$

Note 1.3.5. The function $\tilde{f}$ is defined by

$$\tilde{f}(q) = f\left(\frac{\log q}{2\pi i}\right).$$

Observe that the logarithm is multi-valued, but choosing a different value of the logarithm is the same as adding an integer to $\frac{\log q}{2\pi i}$. The periodicity of f hence implies that $\tilde{f}(q)$ does not depend on the chosen value of the logarithm.

Note 1.3.6. Any weakly modular function can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

for some $a_n \in \mathbb{C}$ where $q = e^{2\pi i z}$; we call this the *q-expansion of f*. This is just the Laurent series of $\tilde{f}$ around $q = 0$, which converges for $0 < |q| < \varepsilon$ for ε sufficiently small ($\Leftrightarrow \Im(z) \gg 0$)

Definition 1.3.7.

- We say that f is meromorphic at ∞ if $a_n = 0$ for $n < -N$ and some $N \in \mathbb{N}$.
- We say that f is holomorphic at ∞ if $a_n = 0$ for $n < 0$. In this case, we define the value of f at ∞ to be $f(\infty) = \tilde{f}(0) = a_0$.

Definition 1.3.8. Let f be a weakly modular function of weight k and level 1.

1. If f is meromorphic on $\mathcal{H} \cup \{\infty\}$ we say f is a **modular function** (of weight k and level 1).
2. If f is holomorphic on $\mathcal{H} \cup \{\infty\}$ we say f is a **modular form** (of weight k and level 1).
3. If f is holomorphic on $\mathcal{H} \cup \{\infty\}$ and $f(\infty) = 0$ we say f is a **cuspidal modular form** or **cusp form**.

Note 1.3.9. If f and g are modular forms (resp. modular functions) of level 1 and weights k and ℓ, then the product fg is a modular form (resp. modular function) of weight $k + \ell$.

1.4 Eisenstein series

Definition 1.4.1. Let $k \geq 4$ even. Define the **Eisenstein series of weight** k to be the function $G_k \colon \mathcal{H} \to \mathbb{C}$ given by

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(mz+n)^k}. \tag{1.1}$$

Recall the following result from complex analysis:

Proposition 1.4.2. *Let U be an open subset of $\mathbb{C}$, and let $(f_n)_n \geq 0$ be a sequence of holomorphic functions on U that converges uniformly on compact subsets of U. Then the limit function $U \to \mathbb{C}$ is holomorphic.*

Lemma 1.4.3. *The series defining $G_k(z)$ converges absolutely and uniformly on subsets of $\mathcal{H}$ of the form*

$$R_{r,s} = \{x + iy : |x| \le r, \, y \ge s\}.$$

It hence converges to a holomorphic function on $\mathcal{H}$.

Proof. Let $z = x + iy \in R_{r,s}$. We have

$$|mz + n|^2 = (mx + n)^2 + m^2 y^2 \ge (mx + n)^2 + m^2 s^2.$$

For fixed m and n, we distinguish the cases $|n| \le 2r|m|$ and $|n| \ge 2r|m|$. In the first case, we have

$$|mz + n|^2 \ge m^2 s^2 \ge \frac{s^2}{2} m^2 + \frac{s^2}{2(2r)^2} n^2 \ge \min\left\{\frac{s^2}{2}, \frac{s^2}{8r^2}\right\} \cdot (m^2 + n^2).$$

In the second case, the triangle inequality implies

$$|mz + n|^2 \ge (|mx| - |n|)^2 + m^2 s^2 \ge \left(\frac{|n|}{2}\right)^2 + m^2 s^2 \ge \min\left\{\frac{1}{4}, s^2\right\} \cdot (m^2 + n^2).$$

Combining both cases and putting

$$c = \min\left\{\frac{s^2}{2}, \frac{s^2}{8r^2}, \frac{1}{4}, s^2\right\},$$

we get the inequality

$$|mz + n| \ge c^{1/2} (m^2 + n^2)^{1/2} \quad \text{for all } m, n \in \mathbb{Z}, \; z \in R_{r,s}.$$

Hence for all $z \in R_{r,s}$, we have

$$G_k(z) \le \frac{1}{c^{k/2}} \sum_{(m,n) \ne (0,0)} \frac{1}{(m^2 + n^2)^{k/2}}.$$

We rearrange the sum by grouping together, for each fixed $j = 1, 2, 3, \ldots$, all pairs (m, n) with $\max\{|m|, |n|\} = j$. We note that for each j there are $8j$ such pairs (m, n), each of which satisfies

$$j^2 \le m^2 + n^2.$$

Hence

$$|G_k(z)| \le \frac{1}{c^{k/2}} \sum_{j=1}^{\infty} \frac{8j}{j^k} = \frac{8}{c^{k/2}} \sum_{j=1}^{\infty} \frac{1}{j^{k-1}},$$

which is finite and independent of $z \in R_{r,s}$, so $G_k(z)$ converges absolutely and uniformly on $R_{r,s}$. Since every compact subset of $\mathcal{H}$ is contained in some $R_{r,s}$, this finishes the proof by Proposition 1.4.2. □

Remark 1.4.4. This proof clearly fails for $k=2$. One can show that for $k=2$, the series (1.1) is conditionally but not absolutely convergent. We will come back to this issue later in the course.

Proposition 1.4.5. *For every even integer $k \geq 4$, the function G_k is a modular form of weight k and level 1. The q-expansion of G_k is given by*

$$G_k(z) = 2\,\zeta(k) + \frac{2\cdot(2\pi i)^k}{(k-1)!}\cdot\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$ (the Riemann zeta function) and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Proof. One easily checks that $G_k(z+1) = G_k(z)$. Moreover, we have

$$\begin{aligned} G_k\left(-\frac{1}{z}\right) &= \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(m(-\frac{1}{z})+n)^k} \\ &= z^k \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(-m+nz)^k} \\ &= z^k\, G_k(z). \end{aligned}$$

Hence $G_k|_kS = G_k$ and $G_k|_kT = G_k$, so $G_k|_k\alpha = G_k$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ by Theorem 1.2.2 (a). Thus G_k is a weakly modular function of weight k and level 1.

It remains to show that G_k is holomorphic at ∞. Therefore we will determine the q-expansion of G_k. Consider the formula $\sum_{n\in\mathbb{Z}}\frac{1}{z+n} = \pi\cdot\cot(\pi z)$. Thus we obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{z+n} = \pi\cdot\cot(\pi z) = i\pi\left(\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right) = i\pi\left(1+\frac{2}{q-1}\right) = i\pi - 2\pi i\sum_{n=0}^{\infty} q^n,$$

where $q = e^{2\pi i z}$. Differentiating $(k-1)$ times with respect to z, and using that $\frac{\partial}{\partial z} = 2\pi i q\frac{\partial}{\partial q}$, leads to

$$\begin{aligned} \sum_{n\in\mathbb{Z}}\frac{-(k-1)!}{(z+n)^k} &= \frac{\partial^{k-1}}{\partial z^{k-1}}\left(i\pi - 2\pi i\sum_{n=0}^{\infty} q^n\right) \\ &= -2\pi i\sum_{n=1}^{\infty}(2\pi i n)^{k-1}q^n \\ &= -(2\pi i)^k\sum_{n=1}^{\infty} n^{k-1}q^n \end{aligned}$$

(We are using here that k is even; for k odd we get an additional $-$ sign.)

Hence we get

$$t_k(z) := \sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}e^{2\pi i n z}.$$

Now we can split up the original sum of the function G_k into two parts, one where $m = 0$ and one where $m \neq 0$. Afterwards we will simplify both parts using symmetry (remember again that k is even) of the sums and the above formula:

$$\begin{aligned} G_k(z) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^k} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} \\ &= 2\sum_{n=1}^{\infty} \frac{1}{n^k} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k) + 2\sum_{m=1}^{\infty} t_k(mz) \\ &= 2\zeta(k) + 2\sum_{m=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z} \\ &= 2\zeta(k) + \frac{2\cdot(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1} q^{nm} \end{aligned}$$

From there we obtain the proposed q-expansion by resorting the last sum:

$$G_k(z) = 2\zeta(k) + \frac{2\cdot(2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} \underbrace{\sum_{d|l} d^{k-1}}_{\sigma_{k-1}(l)} q^l$$

And since G_k has a q-expansion without any negative powers of q, G_k is holomorphic at ∞. Thus G_k is indeed a modular form. $\square$

Definition 1.4.6. The Bernoulli numbers are the rational numbers B_k, for $k \geq 0$, defined by the equation

$$\frac{t}{\exp(t) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \in \mathbb{Q}[[t]].$$

Remark 1.4.7. The Bernoulli numbers are of great importance in mathematics. Barry Mazur once said: *"When a Bernoulli number sneezes, the tremors can be felt in all of mathematics."*

Lemma 1.4.8. *We have*

$$B_k \neq 0 \qquad \Leftrightarrow \qquad k = 1 \text{ or } k \text{ is even.}$$

Proof. Exercise sheet 2. $\square$

Example 1.4.9. The first few non-zero Bernoulli numbers

$$B_0 = 0, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{3}, \quad B_6 = \frac{1}{42}, \\ B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

Lemma 1.4.10. *If $k \geq 2$ is an even integer, then*

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}.$$

Proof. Exercise sheet 2. □

Definition 1.4.11. Let $k \geq 4$ be even. The normalised **Eisenstein series** of weight k is given by

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

1.5 The valence formula

Definition 1.5.1. Let $f \neq 0$ be a meromorphic function $\mathcal{H} \to \mathbb{C}$ and let $P \in \mathcal{H}$. The unique integer n such that $(z-P)^{-n} f(z)$ is holomorphic and non-vanishing at P is called the **order of** f **at** P and denoted by $v_P(f)$. We say f has a **zero of order** n **at** P if n is positive, and f has a **pole of order** n **at** P if n is negative.

Definition 1.5.2. Consider the Laurent expansion of f around P

$$f(z) = \sum_{n \geq n_0} c_n (z-P)^n.$$

Then the **residue of** f **at** P is $\mathrm{Res}_P(f) = c_{-1} \in \mathbb{C}$.

Lemma 1.5.3. *If f is meromorphic around a point P, then*

$$\mathrm{Res}_P(f/f') = v_P(f).$$

Proof. Exercise. □

We recall without proof the following results from complex analysis:

Theorem 1.5.4. *(Cauchy's integral formula) Let g be a holomorphic function on an open subset $U \subseteq \mathbb{C}$ and let C be a contour in U. Then for each $P \in U$, we have*

$$\int_C \frac{g(z)}{z-P} dz = 2\pi i \cdot g(P).$$

Corollary 1.5.5. *Let $C(P, r, \alpha)$ be an arc of a circle of radius r and angle α around a point P. If g is holomorphic at P, then*

$$\lim_{r \to 0} \int_{C(P,r,\alpha)} \frac{g(z)}{z-P} dz = \alpha i \cdot g(P).$$

(Here, we integrate counterclockwise.)

The following result relates the contour integral of the logarithmic derivative of f to the orders of f at the interior points:

Theorem 1.5.6. *(Argument principle) Let f be a meromorphic function on an open subset $U \subseteq \mathbb{C}$, and let C be a contour in U not passing through any zeros or poles of f. Then*

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \mathrm{int}(C)} v_P(f).$$

Note 1.5.7. By Lemma 1.5.3, we have

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \mathrm{int}(C)} \mathrm{Res}_P(f'/f). \tag{1.2}$$

Corollary 1.5.8. *Let $C(P, r, \alpha)$ be an arc of a circle of radius r and angle α around a point P. If f is meromorphic at P, then*

$$\lim_{r \to 0} \int_{C(P,r,\alpha)} \frac{f'(z)}{f(z)} dz = \alpha i \cdot v_P(f).$$

Now assume that f is a weakly modular funktion (of weight k and level 1).

Remark 1.5.9. Since $f|_k \alpha = f$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, we have $v_{\alpha P}(f) = v_p(f)$. Hence $v_P(f)$ is well-defined for P being a $\mathrm{SL}_2(\mathbb{Z})$ orbit in $\mathcal{H}$.

Moreover, if f is meromorphic at ∞, we can define the order of f at ∞ by

$$v_\infty(f) := v_0(\tilde{f}).$$

The following theorem is fundamental for studying the spaces of modular forms:

Theorem 1.5.10. *(The valence formula) Let $f \neq 0$ be a modularfunction of weight k and level 1. Then f has finitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits of zeros and poles in $\mathcal{H}$, and*

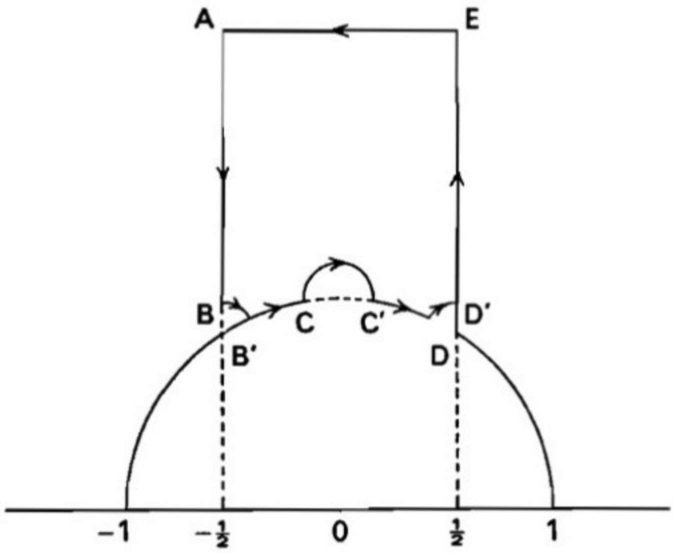
$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{P \in W} v_P(f) = \frac{k}{12}, \tag{1.3}$$

where $\rho = e^{2\pi i/3}$ and W is the set of all $\mathrm{SL}_2(\mathbb{Z})$-orbits in $\mathcal{H}$ except the orbits of i and ρ.

Proof. Recall the fundamental domain from 1.2.2 and let $\mathcal{C}$ be the contour as shown in the figure below. Here $\Im(A) = \Im(E) = R$ (we will later let $R \to +\infty$) and the three small circles have radius r. We assume that R is sufficiently large and r sufficiently small that the interior of $\mathcal{C}$ contains all the zeros and poles of f except those at i, ρ, $\rho + 1$ and ∞.

Simplifying assumption: We assume for simplicity f has no zeros or poles on the boundary of the fundamental domain, except possibly at i and ρ. (In the case where it does contain zeros or poles of f, the contour has to be modified using additional small arcs going around these zeros or poles in the counterclockwise direction.)

We will now calculate $\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz$ in two different ways and compose the results afterwards.



(1) Computing the integral using Theorem 1.5.6, we get

$$\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{P \in \text{interior}(\mathcal{C})} v_P(f) = 2\pi i \sum_{P \in W} v_P(f),$$

where W is the set described in the stated theorem. The last equality is satisfied by the simplifying assumption, so the interior of the fundamental domain contains exactly one representative of every pole or zero $\mathrm{SL}_2(\mathbb{Z})$-orbit of $\mathcal{H}$.

(2) Secondly, we estimate the integral by splitting up the contour in 8 parts. Let $\mathcal{C}_1$ be the part from E to A, $\mathcal{C}_2$ be the part from A to B, and so on, such that in the end $\mathcal{C}_8$ is the part from D' to E.

(i) Note that since f is a modular function, we have $f(z) = f(z+1)$. Hence also $f'(z) = f'(z+1)$, and we have

$$\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)} dz = \int_{\mathcal{C}_2} \frac{f'(z+1)}{f(z+1)} dz = -\int_{\mathcal{C}_8} \frac{f'(z)}{f(z)} dz,$$

so

$$\int_{\mathcal{C}_2} \frac{f'(z)}{f(z)} dz + \int_{\mathcal{C}_8} \frac{f'(z)}{f(z)} dz = 0.$$

(ii) Now we consider $\mathcal{C}_1$ and change the variable by $q(z) = e^{2\pi i z}$. This maps $\mathcal{C}_1$ to a clockwise oriented circle around the origin with radius $e^{-2\pi R}$. Furthermore we have $f(z) = \tilde{f}(q(z))$, thus $f'(z) = \tilde{f}'(q(z))\, q'(z)$ and since f is a modular function, $\tilde{f}$ is meromorphic at 0. Therefore

$$\begin{aligned}
\int_{\mathcal{C}_1} \frac{f'(z)}{f(z)} dz &= \int_{\mathcal{C}_1} \frac{\tilde{f}'(q(z))q'(z)}{\tilde{f}(q(z))} dz \\
&= \int_{q(\mathcal{C}_1)} \frac{\tilde{f}'(q)}{\tilde{f}(q)} dq \\
&= -2\pi i \operatorname{Res}_0 \left(\frac{\tilde{f}'}{\tilde{f}} \right) \\
&= -2\pi i \ v_0(\tilde{f}) \\
&= -2\pi i \ v_\infty(f).
\end{aligned}$$

(iii) $\mathcal{C}_5$ is half of a circle around i. We deduce from Corollary 1.5.8 that

$$\lim_{r \to 0} \int_{\mathcal{C}_5} \frac{f'(z)}{f(z)} dz = -\frac{1}{2} \ 2\pi i \ v_i(f).$$

Similarly we get

$$\begin{aligned}
\lim_{r \to 0} \int_{\mathcal{C}_3} \frac{f'(z)}{f(z)} dz &= -\frac{1}{6} \ 2\pi i \ v_\rho(f) \\
\lim_{r \to 0} \int_{\mathcal{C}_7} \frac{f'(z)}{f(z)} dz &= -\frac{1}{6} \ 2\pi i \ v_{\rho+1}(f) = -\frac{1}{6} \ 2\pi i \ v_\rho(f).
\end{aligned}$$

(iv) So it remains to study $\mathcal{C}_4$ and $\mathcal{C}_6$. Therefore consider $u(z) = -\frac{1}{z}$. This maps $\mathcal{C}_6$ to $-\mathcal{C}_4$ and we have $f(z) = z^{-k} f(u(z))$, hence

$$f'(z) = -kz^{-k-1} f(u(z)) + z^{-k} f'(u(z))u'(z).$$

So

$$\begin{aligned}
\int_{\mathcal{C}_4} \frac{f'(z)}{f(z)} dz &= \int_{\mathcal{C}_4} \frac{-k}{z} dz + \int_{\mathcal{C}_4} \frac{f'(u(z))u'(z)}{f(u(z))} dz \\
&= \frac{2\pi i k}{12} + \int_{u(\mathcal{C}_4)} \frac{f'(u)}{f(u)} du \\
&= \frac{2\pi i k}{12} - \int_{\mathcal{C}_6} \frac{f'(u)}{f(u)} du
\end{aligned}$$

and thus

$$\int_{\mathcal{C}_4} \frac{f'(z)}{f(z)} dz + \int_{\mathcal{C}_6} \frac{f'(z)}{f(z)} dz = 2\pi i \frac{k}{12}.$$

Composing (i) to (iv) yields

$$\int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = 2\pi i \left(\frac{k}{12} - \frac{1}{3} v_\rho(f) - \frac{1}{2} v_i(f) - v_\infty(f) \right).$$

Combining this with the result in (1) gives us exactly the proposed formula. □

1.6 Applications to modular forms

The valence formula provides some interesting consequences to spaces of modular forms which we will investigate below.

Definition 1.6.1. Let M_k be the set of all modular forms of weight k and level 1 and let S_k be the set of all cusp forms of weight k and level 1.

Remark 1.6.2. It can be easily checked that these are both vector spaces over $\mathbb{C}$.

Lemma 1.6.3.

(a) $M_k = \{0\}$ *for* $k < 0$ *and* $k = 2$.

(b) $S_k = \{0\}$ *for* $k < 12$.

(c) M_0 *is the set of all constant functions* $\mathcal{H} \to \mathbb{C}$ *and thus isomorphic to* $\mathbb{C}$.

Proof. (a) Let $f \in M_k$, $f \neq 0$. Then $v_z(f) \geq 0$ for all $z \in \mathcal{H} \cup \{\infty\}$. So by the valence formula we get $k \geq 0$. Moreover a sum of non-negative integer multiples of $\frac{1}{2}$ and $\frac{1}{3}$ can't equal $\frac{1}{6}$. Thus $k \neq 2$.

(b) Let $f \in S_k$, $f \neq 0$. Then $v_\infty(f) \geq 1$, hence $k \geq 12$ by valence formula.

(c) Let $f \in M_0$. Then the constant function $g := f(\infty)$ is also in M_0, so $f - g \in S_0$ and therefore $f = g$ since $S_0 = \{0\}$.

$\square$

Definition 1.6.4. Define

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

Remark 1.6.5. In the prologue of this lecture we defined $\Delta = q \cdot \prod_{n \in \mathbb{N}} (1 - q^n)^{24}$. We will prove later that this is indeed the same Δ as the one in Definition 1.6.4.

Note 1.6.6. Since E_4 and E_6 are modular forms of weight 4 and 6, respectively, Δ is a modular form of weight 12. Since the q-expansion has zero constant coefficient, it is indeed a cusp form.

Lemma 1.6.7. *The modular form* Δ *has a simple zero at* ∞ *and no other zeros.*

Proof. Using the known q-expansions of E_4 and E_6, one can compute the q-expansion of Δ as

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots,$$

so Δ has a simple zero at ∞. Now since Δ is a modular form, all the quantities $v_\star(\Delta)$ occurring in Theorem 1.5.10 are non-negative, so the only way to get equality is if there are no zeros apart from the one at ∞. $\square$

Proposition 1.6.8. *S_{12} is one-dimensional over $\mathbb{C}$ and spanned by Δ.*

Proof. Let $f \in S_{12}$ and define a function g by

$$g(z) = f(z) - \frac{f(i)}{\Delta(i)}\Delta(z).$$

This function is well-defined since Δ does not vanish on $\mathcal{H}$, so $\Delta(i) \neq 0$. Clearly $g \in S_{12}$ and $g(i) = 0$. Using the valence formula yields

$$v_\infty(g) + \frac{1}{2}v_i(g) + \frac{1}{3}v_\rho(g) + \sum_{p\in W} v_p(g) = 1.$$

But this is a contradiction since $v_\infty(g) \geq 1$ and $v_i(g) \geq 1$. Therefore g has to be zero and

$$f = \frac{f(i)}{\Delta(i)}\Delta \in \mathbb{C} \cdot \Delta.$$

□

Corollary 1.6.9.

1. *For all $k \in \mathbb{Z}$, the map*
$$M_k \to S_{k+12}, \ f \mapsto f \cdot \Delta$$
is an isomorphism.

2. *For $k \geq 4$ we have $M_k = S_k \oplus (\mathbb{C} \cdot E_k)$.*

Proof. The first statement is trivial for $k < 0$ since then $M_k = S_{k+12} = \{0\}$ by Lemma 1.6.3 (a), (b). So let $k \geq 0$. As Δ is non-vanishing the given map is clearly an injection. Now let $g \in S_{k+12}$. Then $\frac{g}{\Delta}$ is weakly modular of weight $(k + 12) - 12 = k$ and holomorphic on $\mathcal{H}$ since Δ is non-vanishing. Furthermore $v_\infty(g) \geq 1$ by assumption, so

$$v_\infty\left(\frac{g}{\Delta}\right) = v_\infty(g) - v_\infty(\Delta) = v_\infty(g) - 1 \geq 0.$$

So $\frac{g}{\Delta} \in M_k$. Therefore the given map is also onto, thus bijectiv.

For the second part of the corollary we just have to note that S_k is the kernel of the linear map $M_k \to \mathbb{C}, \ f \mapsto f(\infty)$. Thus we have $\dim(M_k/S_k) \leq 1$. On the other hand we know that $E_k \in M_k \setminus S_K$ since $E_k(\infty) \neq 0$. So $M_k = S_k \oplus (\mathbb{C}\ E_k)$. □

Theorem 1.6.10.

(a) The space M_k is finite dimensional over $\mathbb{C}$ for all $k \in \mathbb{Z}$.

(b) Let $k \geq 0$ even. Then

$$\dim(M_k) = \begin{cases} 1 + \left\lfloor \frac{k}{12} \right\rfloor, & k \neq 2 \mod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor, & k = 2 \mod 12. \end{cases}$$

Otherwise $M_k = \{0\}$.

(c) A basis for M_k is given by $\{E_4^a E_6^b\colon a,b \in \mathbb{N}_0, 4a+6b=k\}$.

Proof. (a) This is a consequence of part (b).

(b) We will prove this by induction on k. First of all note that the statement is clear for odd k since there aren't any nonzero weakly modular functions of odd weight. Moreover we already know that $\dim(M_0) = 1$, $\dim(M_2) = 0$ and $\dim(M_k) = 0$ for $k < 0$ by Lemma 1.6.3 (a) and (c). In addition we have $\dim(M_k) = 1$ for $k = 4, \ldots, 10$ since $\dim(M_k) = \dim(S_k) + 1$ by Corollary 1.6.9 and $S_k = \{0\}$ for these k's by Lemma 1.6.3 (b). Hence the statement is true for $k = 0, \ldots, 10$.

Let now $k \geq 12$. Then

$$\dim(M_k) = \dim(M_{k-12}) + 1$$

since $\dim(S_k) = \dim(M_{k-12})$ by Corollary 1.6.9. So the statement is true for all k by induction in steps of 12.

(c) We will use again induction to prove the statement. Note that there is nothing to show for odd k, $k < 0$ and $k = 2$ since in these cases $M_k = \{0\}$. The case $k = 0$ is also trivial because M_0 is the set of all constant functions, hence generated by $1 = E_4^0 E_6^0$.

Let now $k \geq 4$ be even. Obviously there is always a pair (a, b) such that $a, b \in \mathbb{Z}_{\geq 0}$ and $4a+6b = k$. Pick such a pair. Let $f \in M_k$. Then f can be written in the form

$$f = \lambda\ E_4^a E_6^b + g$$

for some $\lambda \in \mathbb{C}$ and $g \in S_k$ since the modular form $E_4^a E_6^b$ is in M_k and does not vanish at infinity. So there is an $h \in M_{k-12}$ such that $g = h \cdot \Delta$ by corollary 1.6.9 and by induction we may assume h to be a linear combination of $E_4^r E_6^s$ where $r, s \in \mathbb{Z}_{\geq 0}$ and $4r + 6s = k - 12$. Hence

$$h \cdot \Delta = h \cdot \left(\frac{E_4^3 - E_6^2}{1726} \right)$$

is a linear combination of $E_4^{r+3} E_6^s$ and $E_4^r E_6^{s+2}$ and since

$$4(r+3) + 6s = 4r + 6(s+2) = k$$

the function h is a linear combination of $E_4^p E_6^q$ with $4p + 6q = k$. So the linear span of these functions contains g and hence also f. Therefore

$$M_k = \operatorname{span}\{E_4^a E_6^b\colon a, b \in \mathbb{N}_0, 4a + 6b = k\}.$$

To show that the given set is indeed a basis of M_k it suffices to check that

$$\left|\left\{(a,b) \in \mathbb{Z}_{\geq 0}^2\colon 4a + 6b = k\right\}\right| = \dim(M_k).$$

This can again be easily seen by induction in steps of 12 (exercise).

□

Example 1.6.11. For the first few values of k, the dimensions of M_k and S_k are given by

k	$\dim M_k$	$\dim S_k$
0	1	0
2	0	0
4	1	0
6	1	0
8	1	0
10	1	0
12	2	1
14	1	0
16	2	1

Example 1.6.12. Both, E_4^2 and E_8 are in M_8. But $\dim(M_8) = 1$ by Theorem 1.6.10 (b). Hence E_4^2 and E_8 are linearly dependent and as both are 1 at infinity, we can conclude that E_4^2 and E_8 are equal. So

$$\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^2 = E_4^2 = E_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

, so

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

This is very hard to prove (or even conjecture!) without using the theory of modular forms.

Example 1.6.13. From the theorem, we deduce that

$$M_{30} = \mathbb{C}E_{30} \oplus \mathbb{C}\Delta E_{18} \oplus \mathbb{C}\Delta^2 E_6.$$

I claim that another basis for the same space is given by

$$M_{30} = \mathbb{C}E_6^5 \oplus \mathbb{C}\Delta E_6^3 \oplus \mathbb{C}\Delta^2 E_6^2.$$

Note that these forms are linearly independent (exercise), so since $\dim(M_{30}) = 3$, they form a basis.

The following theorem is a very useful consequence of the fact that the spaces of modular forms are finite-dimensional:

Theorem 1.6.14. *Let f be a modular form of weight k and level 1 with q-expansion $\sum_{n=0}^{\infty} a_n q^n$. Suppose that*

$$a_j = 0 \qquad \text{for all } j = 0, \ldots, \lfloor k/12 \rfloor.$$

Then $f = 0$.

Proof. Suppose that $f \neq 0$. Then the hypothesis implies that

$$v_\infty(f) \geq \lfloor k/12 \rfloor + 1 > k/12.$$

Hence the left-hand side of (1.3) is strictly greater than $k/12$, which gives a contradiction. □

Corollary 1.6.15. *Let f, g be modular forms of the same weight k and level 1, with q-expansions $\sum_{n=0}^{\infty} a_n q^n$ and $\sum_{n=0}^{\infty} b_n q^n$, respectively. Suppose that*

$$a_j = b_j \qquad \text{for all } j = 0, ..., \lfloor k/12 \rfloor.$$

Then $f = g$.

Corollary 1.6.15 is a very powerful tool: it allows us to conclude that two modular forms are identical if we only know a priori that their q-expansions agree to a certain finite precision.

1.7 The q-expansion of Δ

The aim of this section is to prove the product formula for the q-expansion of Δ. We start with the following definition:

Definition 1.7.1. We define

$$G_2(z) = \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}, (m,n) \neq 0} \frac{1}{(mz+n)^2} \right)$$

and $E_2(z) = \frac{3}{\pi^2} \cdot G_2(z)$.

Lemma 1.7.2.

1. *The series in Definition 1.7.1 is convergent, but not absolutely convergent, and defines a holomorphic function on $\mathcal{H}$*[1]*.*

2. *We have*

$$G_2(z) = 2\zeta(2) - 8\pi \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

Proof. 1. Exercise.
2. Argue as in the proof of proposition 1.4.5. □

Proposition 1.7.3. *The functions G_2 and E_2 satisfies the transformation property*

$$z^{-2} G_2\left(-\frac{1}{z}\right) = G_2(z) - 2\pi i z, \tag{1.4}$$

$$z^{-2} E_2\left(-\frac{1}{z}\right) = E_2(z) - \frac{6i}{\pi z}. \tag{1.5}$$

[1]It is not a modular form, however: it can't be, since $M_2 = \{0\}$.

The proof of this result is based on the following lemma, which gives an example of two double series that contain the same terms but sum to different values due to the order of summation being different.

Lemma 1.7.4. *For all $z \in \mathcal{H}$, we have*

$$\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=0, \tag{1.6}$$

$$\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=-\frac{2\pi i}{z}. \tag{1.7}$$

Proof. We start with the sum

$$\sum_{-N\leq n<N}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=\frac{1}{mz-N}-\frac{1}{mz+N}.$$

Using this, we compute the inner sum of (1.6) as

$$\sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=\lim_{N\to\infty}\sum_{-N\leq n<N}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right) \tag{1.8}$$

$$=\lim_{N\to\infty}\frac{1}{mz-N}-\frac{1}{mz+N}. \tag{1.9}$$

$$=0, \tag{1.10}$$

which implies (1.6).

The proof of the second formula is more complicated, and I will not give the proof here. For a reference, see Serre's "A course in Arithmetic". □

We can now prove Proposition 1.7.3:

Proof. Recall that

$$G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}.$$

Subtracting (1.6) and simplifying, we obtain the alternative expression

$$G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2(mz+n+1)}. \tag{1.11}$$

Also, we have

$$z^{-2}G_2(-1/z)=2\zeta(2)z^{-2}+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(nz-m)^2} \tag{1.12}$$

$$=2\zeta(2)+\sum_{m\in\mathbb{Z}}\sum_{n\neq 0}\frac{1}{(nz-m)^2} \tag{1.13}$$

$$=2\zeta(2)+\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{(mz+n)^2}; \tag{1.14}$$

note that in the second equality we just relabelled the parameters, but did not change the order of summation.

Subtracting (1.7) and simplifying, we obtain

$$z^{-2}G_2(-1/z) + \frac{2\pi i}{z} = 2\zeta(2) + \sum_{n\in\mathbb{Z}}\sum_{m\neq 0} \frac{1}{(mz+n)^2(mz+n+1)}, \tag{1.15}$$

and by imitating the proof of Lemma 1.4.3 one can show that the sum on the right-hand side is absolutely convergent. We can hence change the order of summation, and we see that (1.15) is equal to (1.11). □

Corollary 1.7.5. *The q-expansion of Δ is given by*

$$\Delta = q\prod_{n\geq 1}(1-q^n)^{24}.$$

Proof. Let $D(z) = q\prod_{n\geq 1}(1-q^n)^{24}$.

Let $D(z) = q \cdot \prod_{n=1}^{\infty}(1-q^n)^{24}$ where $q = e^{2\pi iz}$ as usual. We can check that this product converges sufficiently fast for D to be defined and holomorphic on $\mathcal{H}$. Evidently $D(z+1) = D(z)$ and $D(z) \to 0$ as $\Im(z) \to \infty$. So to check that it is a modular form of weight 12 (clearly cuspidal), it suffices to show that $D(-\frac{1}{z}) = z^{12}D(z)$. The result then follow immediately, since we already know that S_{12} is 1-dimensional.

Recall that $\frac{\partial d}{\partial z} = 2\pi i q\frac{\partial}{\partial q}$. Then

$$\begin{aligned}
\frac{\partial}{\partial z}(\log(D(z))) &= \frac{\partial}{\partial z}\left(\log(q) + \sum_{n=1}^{\infty} 24\log(1-q^n)\right)\\
&= 2\pi i + 24\sum_{n=1}^{\infty}\frac{-2\pi i n q^n}{1-q^n}\\
&= 2\pi i\left(1 - 24\sum_{n=1}^{\infty} nq^n\sum_{r=0}^{\infty} q^r\right)\\
&= 2\pi i\left(1 - 24\sum_{n=1}^{\infty}\sum_{r=0}^{\infty} nq^{nr}\right)\\
&= 2\pi i\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\\
&= 2\pi i\ E_2(z).
\end{aligned}$$

Hence finally

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\log\left(\frac{D(-1/z)}{z^{12}D(z)}\right)\right) &= \frac{1}{z^2}2\pi i\ E_2\left(-\frac{1}{z}\right) - \frac{12}{z} - 2\pi i\ E_2(z)\\
&= \frac{2\pi i}{z^2}\left(E_2\left(-\frac{1}{z}\right) - \left(z^2E_2(z) + \frac{6z}{i\pi}\right)\right)\\
&= 0.
\end{aligned}$$

So there is a constant λ sucht that $D(-\frac{1}{z}) = \lambda z^{12} D(z)$ for all $z \in \mathcal{H}$. For $z = i$ solves this to $D(i) = D(-\frac{1}{i}) = \lambda D(i)$, and since $D(i) \neq 0$ we have $\lambda = 1$, and therefore $D(-\frac{1}{z}) = z^{12} D(z)$. □

We can now expand the product formula for $\Delta(z)$ as

$$\Delta(z) = \sum_{n \geq 1} \tau(n) q^n \qquad \text{for some } \tau(n) \in \mathbb{Z}.$$

Conjecture 1.7.6. *(Ramanujan, 1916)*

1. *For m, n coprime, we have $\tau(mn) = \tau(m)\tau(n)$.*

2. *For p prime and $n > 0$, we have*

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^1 1 \tau(p^{n-1}).$$

3. *We have $|\tau(p)| \leq 2p^{\frac{11}{2}}$ for all primes p.*

We will see a proof of properties 1) and 2) later in the course, in the section on Hecke operators. Property 3) was proved by Deligne in 1974 as a consequence of his proof of the Weil conjectures, for which he was awarded the Fields medal in 1978.

2 Modular forms of higher level

The idea is to look at functions transforming nicely under subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

2.1 Congruence subgroups

Definition 2.1.1. For $N \in \mathbb{N}$ define the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

We will call this group the **principal congruence subgroup of level** N.

Note 2.1.2. $\Gamma(N)$ is the kernel of the group homomorphism induced by the reduction map $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$:

$$\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

It is hence a normal subgroup of finite index. (Ex: show that π_N is sujective. This statement goes by the name of "strong approximation for SL_2". It can be shown to be false for $\mathrm{GL}_2(\mathbb{Z})$.)

Definition 2.1.3. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a **congruence subgroup** if there exists $N \geq 1$ such that $\Gamma(N) \subseteq \Gamma$. The least such N is called the **level** of Γ.

Lemma 2.1.4. *Any congruence subgroup has finite index in* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. It sufficies to show that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] < \infty$ for all $N \in \mathbb{N}$. But this is clear as $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is finite. □

Remark 2.1.5. The converse to Lemma 2.1.4 is false. There exist finite index $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ which don't contain $\Gamma(N)$ for any N. (For example there is one of index 7.) But every finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ is congruence for $n \geq 3$. So SL_2 is quite unusual. (Bass-Serre-Milnor theorem)

Definition 2.1.6. Other standard congruence subgroups of level N are given by

- $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$,
- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$.

Note 2.1.7. We have a chain of inclusions

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

These inclusions are in general strict; however, all of them are equalities for $N = 1$, and $\Gamma_0(2) = \Gamma_1(2)$.

Lemma 2.1.8. *For $N \geq 1$, we have*

$$[\Gamma_1(N) : \Gamma(N)] = N, \quad [\Gamma_0(M) : \Gamma_1(N)] = N \prod_{p|N} \left(1 - \frac{1}{p}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Definition 2.1.9. Let Γ be a congruence subgroup. We say that Γ is even (resp. odd) if $-\mathrm{Id} \in \Gamma$ (resp. $\mathrm{Id} \notin \Gamma$). WE define the projective index of Γ to be

$$d_\Gamma = [\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}],$$

where $\bar{\Gamma}$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$.

2.2 Fundamental doins and cusps

Proposition 2.2.1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let R be a set of coset representatives for the quotient $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Then the set*

$$D_\Gamma = \bigcup_{\gamma \in R} \gamma D$$

has the property that for any $z \in \mathcal{H}$ there exists $\gamma \in \Gamma$ such that $\gamma z \in D_\Gamma$. Furthermore, γ is unique up to multiplication by an element of $\Gamma \cap \{\pm\mathrm{Id}\}$, except possibly if γz lies on the boundary of D. We call D_Γ a ***fundamental domain for*** Γ.

Proof. If $z \in \mathcal{H}$, thern there exists $g \in \mathrm{SL}_2(\mathbb{Z})$ and $z_0 \in D$ such that $g.z = z_0$. The coset decomposition implies that we can express g uniquely as $\gamma^{-1}\gamma'$ with $\gamma \in \Gamma$ and $\gamma' \in R$. We now have

$$\gamma.z = \gamma g.z_0 = \gamma'.z_0 \in D_\Gamma.$$

The uniqueness is left as an exercise. □

Example 2.2.2. Let $\Gamma = \Gamma_0(2)$. A system of representatives for the quotient $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} = \{\mathrm{Id}, S, ST\}.$$

Using this, one can draw the fundamental domain for Γ.

Note that there are now two points in its closure which do not belong to $\mathcal{H}$: the cusp ∞, as well as 0.

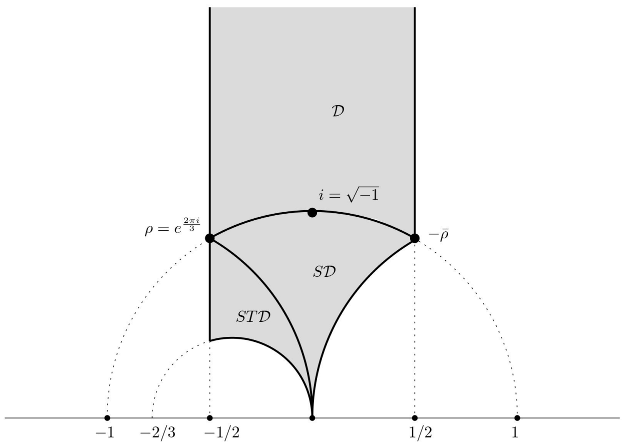


Figure 2.1: A fundamental domain for $\Gamma_0(2)$

Definition 2.2.3. The set $\mathbb{P}^1(\mathbb{Q})$, the **projective line over** $\mathbb{Q}$, consists of $\mathbb{Q} \cup \{\infty\}$. We give this an action of $\mathrm{SL}_2(\mathbb{Z})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .x = \frac{ax+b}{cx+d}$$

where the right-hand-side is interpreted as $\frac{a}{c}$ if $x = \infty$, and as ∞ if $cx + d = 0$.

Proposition 2.2.4. $\mathrm{SL}_2(\mathbb{Z})$ *acts transitively on* $\mathbb{P}^1(\mathbb{Q})$.

Proof. Clearly it sufficies to show that for any $x \in \mathbb{P}^1(\mathbb{Q})$ we can map ∞ to x. For $x = \infty$ we have $\infty.1 = \infty$. So let $x = \frac{a}{c}$ with $a, c \in \mathbb{Z}$ coprime. Then there are $r, s \in \mathbb{Z}$ such that $ar + cs = 1$, thus $\left(\begin{smallmatrix} a & -s \\ c & r \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\left(\begin{smallmatrix} a & -s \\ c & r \end{smallmatrix}\right).\infty = x$. □

Note 2.2.5. An easy computation shows that the stabiliser of ∞ in $\mathrm{SL}_2(\mathbb{Z})$ is the subgroup

$$\mathrm{SL}_2(\mathbb{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

It follows from Proposition 2.2.4 that we hence have a bijection

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) / \mathrm{SL}_2(\mathbb{Z})_\infty &\to \mathbb{P}^1(\mathbb{Q}), \\ \gamma \, \mathrm{SL}_2(\mathbb{Z})_\infty &\mapsto \gamma\infty. \end{aligned}$$

Definition 2.2.6. For $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ of finite index we define **the set of cusps of** Γ, denoted by $\mathrm{Cusps}(\Gamma)$, as the set of Γ-orbits in $\mathbb{P}^1_{\mathbb{Q}}$.

Example 2.2.7. Let p be prime. Then $\mathrm{Cusps}(\Gamma_0(p)) = \{[\infty], [0]\}$.

Proof. Let $\frac{u}{v} \in \mathbb{Q}$ with $u, v \in \mathbb{Z}$ coprime. Then there are $r, s \in \mathbb{Z}$ such that $ur + vs = 1$, so $\left(\begin{smallmatrix} u & -s \\ v & r \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\left(\begin{smallmatrix} u & -s \\ v & r \end{smallmatrix}\right).\infty = \frac{u}{v}$. We will distinguish two cases:

(1) If p divides v then $\left(\begin{smallmatrix} u & -s \\ v & r \end{smallmatrix}\right) \in \Gamma_0(p)$, so $\frac{u}{v} \in [\infty]$. Conversly, if $\gamma \in \Gamma_0(p)$ then p divides the denominator of $\gamma.\infty$ by definition. So the orbit of ∞ is given by all fractions $\frac{u}{v}$ with p dividing the denominator v.

(2) If v is not divisible by p we can note that

$$u(r + \lambda v) + v(s - \lambda u) = 1$$

and since p is not a divisor of v we find $\lambda \in \mathbb{Z}$ such that $r' = r + \lambda v \in p\mathbb{Z}$. Therefore $\left(\begin{smallmatrix} s' & u \\ -r' & v \end{smallmatrix}\right) \in \Gamma_0(p)$ where $s' = s - \lambda u$ and $\left(\begin{smallmatrix} s' & u \\ -r' & v \end{smallmatrix}\right).0 = \frac{u}{v}$ by definition. So $\frac{u}{v} \in [0]$. Conversely, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ then p does not divide d since $ad - bc = 1$. Thus p cannot divide the denominator of $\gamma.0$. Therefore the orbit of 0 is given by all fractions $\frac{u}{v}$ with p not dividing the denominator v.

So this is everything and there are exactly two distinct orbits as claimed. □

Note 2.2.8. By Note 2.2.5, we see that

$$\operatorname{Cusps}(\gamma) = \Gamma \backslash \operatorname{SL}_2(\mathbb{Z}) / \operatorname{SL}_2(\mathbb{Z})_\infty.$$

In particular, we have a sujective map

$$\operatorname{SL}_2(\mathbb{Z}) / \operatorname{SL}_2(\mathbb{Z})_\infty \twoheadrightarrow \operatorname{Cusps}(\Gamma).$$

Definition 2.2.9. If $P = [t] \in \operatorname{Cusps}(\Gamma)$, denote by Γ_t the stabilizer for t in Γ.

Lemma 2.2.10. *Choose* $\gamma_t \in \operatorname{SL}_2(\mathbb{Z})$ *such that* $\gamma_t(\infty) = t$. *Then*

$$\Gamma_t = \Gamma \cap \gamma_t \operatorname{SL}_2(\mathbb{Z})_\infty \gamma_t^{-1}.$$

Proof. Let $h \in \Gamma$. Then

$$\begin{aligned} h \in \Gamma_t \quad &\Leftrightarrow \quad h.t = t \\ &\Leftrightarrow \quad h\gamma_t(\infty) = \gamma_t(\infty) \\ &\Leftrightarrow \quad \gamma_t^{-1} h \gamma_t(\infty) = \infty \\ &\Leftrightarrow \quad \gamma_t^{-1} h \gamma_t \in \operatorname{SL}_2(\mathbb{Z})_\infty \\ &\Leftrightarrow \quad h \in \gamma_t \operatorname{SL}_2(\mathbb{Z})_\infty \gamma_P t{-}1. \end{aligned}$$

□

Note 2.2.11. It follows from the proof that we have an injection

$$\Gamma_t \backslash (\gamma_t^{-1} \operatorname{SL}_2(\mathbb{Z})_\infty \gamma_t) \hookrightarrow \Gamma \backslash \operatorname{SL}_2(\mathbb{Z}),$$

so Γ_t has finite index in $\gamma_t^{-1} \operatorname{SL}_2(\mathbb{Z})_\infty \gamma_t$.

Lemma 2.2.12. *The subgroup*

$$H_P = \gamma_t^{-1}\Gamma\gamma_t \cap \mathrm{SL}_2(\mathbb{Z})_\infty \subseteq \mathrm{SL}_2(\mathbb{Z})$$

does not depend on the choice of representative for P*, and it has finite index in* $\mathrm{SL}_2(\mathbb{Z})_\infty$*.*

Proof. If t' is another representative of P, then there exists $\gamma \in \Gamma$ such that $\gamma.t = t'$, so γ_t is replaced by $\gamma\gamma_{t'}$, and

$$(\gamma\gamma_t)^{-1}\Gamma(\gamma\gamma_t) = \gamma_t^{-1}\Gamma\gamma_t.$$

The statement about finite index follows immediately from Note 2.2.11. □

Lemma 2.2.13. *Let* H *be a subgroup of finite index in* $\mathrm{SL}_2(\mathbb{Z})_\infty$*, and let* h *be the index of* $\pm H$ *in* $\mathrm{SL}_2(\mathbb{Z})_\infty$*. Then* H *is one of the following:*

$$H = \begin{cases} \langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \rangle \\ \left\langle \left(\begin{smallmatrix} -1 & h \\ 0 & -1 \end{smallmatrix}\right) \right\rangle \\ \{\pm \mathrm{Id}\} \times \langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \rangle \end{cases}$$

Proof. Exercise. □

Definition 2.2.14. For $H = H_P$, the integer $h_\Gamma(P) = h$ in Lemma 2.2.13 is called the **width of the cusp** P for Γ. The cusp P is

- **irregular** if $\gamma_P^{-1}\Gamma_P\gamma_P$ is of the form $\left\langle \left(\begin{smallmatrix} -1 & h \\ 0 & -1 \end{smallmatrix}\right) \right\rangle$ (then Γ is necessarily odd),
- **regular** if $\gamma_P^{-1}\Gamma_P\gamma_P$ is of the form $\langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \rangle$ (so Γ is odd), of if $\gamma_P^{-1}\Gamma_P\gamma_P$ is of the form $\{\pm \mathrm{Id}\} \times \langle \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \rangle$ (so Γ is even).

Remark 2.2.15. If Γ is normal in $\mathrm{SL}_2(\mathbb{Z})$, the subgroup H_P does not depend on the cusp P, and hence all the cusps have the same width and regularity.