Theta series and modular forms

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1 Lattices and theta series

We will investigate theta series, which give a neat connection between the theory of lattices and the theory of modular forms. We will first recall the basics of lattice theory, define theta series, and study two cases in which they prove useful: one in number theory, with the proof that any integer can be represented as the sum of 4 squares, and one general result that made possible the recent breakthroughs in solving the sphere-packing problem in dimensions 8 and 24. My references for this presentation are [Ebe13] and [Bru+08].

The setup is as follows: we let $V = \mathbb{R}^n$ be equipped with the usual scalar product and we let $Q(x) := \langle x, x\rangle$, $x \in V$, be the quadratic form associated to it. Note that this could be done in more generality by looking at a quadratic space on an arbitrary field.

We recall that a **lattice** L in V is a discrete subgroup of V that spans V as a vector space over $\mathbb{R}$. In other words, L is a finitely-generated $\mathbb{Z}$-module equipped with a positive definite symmetric bilinear form (the restriction of B to L) that spans V when we extend its scalars to $\mathbb{R}$, or, maybe more simply, it is the $\mathbb{Z}$-span of n $\mathbb{R}$-linearly independent vectors of $\mathbb{R}^n$.

We identify V with its dual $V^* = \mathrm{Hom}(V, \mathbb{R})$ and define the **dual lattice** of L, denoted L^*, by

$$L^* = \mathrm{Hom}(L, \mathbb{Z}) = \{y \in V \mid \langle \ell, y\rangle \in \mathbb{Z} \text{ for all } \ell \in L\}.$$

Note that this is now a lattice in V too.

To conclude this introduction to lattices, we record a fundamental result in the theory of functions on lattices:

Proposition 1.1 (Poisson summation formula). *Let $\mathbb{R}^n$ be equipped with the usual scalar product. Let L be a lattice in $\mathbb{R}^n$ and denote L^* its dual. Finally for a smooth rapidly decreasing function $f : \mathbb{R}^n \to \mathbb{C}$, define*

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi\langle x,y\rangle} dx$$

the Fourier transform of f. Then,

$$\sum_{\ell \in L} f(\ell) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)} \sum_{\tilde{\ell} \in L^*} \mathcal{F}(f)(\tilde{\ell}).$$

See [Ebe13] Theorem 2.3 for a proof.

Considering a lattice L of $\mathbb{R}^n$ on which $Q : x \mapsto \langle x, x\rangle$ is integer-valued, we associate to L the generating function counting the number of vectors of L at which Q takes the value $n \in \mathbb{N}$. We let

$$\Theta_L(z) := \sum_{\ell \in L} q^{\frac{1}{2}Q(\ell)}.$$

It is called the **theta function of** L. It is not hard to check that Θ_L converges absolutely on subsets of $\mathcal{H}$ of the form

$$\{z \in \mathcal{H} \mid \Im(z) \geq y_0 > 0\},$$

which implies that it is holomorphic and well-defined on $\mathcal{H}$. Theta series turn out to be an important source of modular forms. We now turn to our first example of the usefulness of theta series.

2 The Jacobi theta series

We first consider the case $n = \dim(V) = 1$ and the simplest example of a unary (one-variable) theta series. It is called the Jacobi theta function and corresponds to the quadratic form $x \mapsto x^2$ evaluated on the lattice $\mathbb{Z} \subset \mathbb{R}$:

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 + 2q + 2q^2 + \dots,$$

where $z \in \mathcal{H}$. This very basic example already has interesting properties and turns out to have surprising applications. We let the reader refer to [Bru+08] pp. 24–27 for more details.

Indeed, the Jacobi theta has the following transformation properties:

Proposition 2.1. *For any $z \in \mathcal{H}$, we have*

$$\theta(z+1) = \theta(z), \quad \theta\left(\frac{-1}{4z}\right) = \sqrt{\frac{2z}{i}}\theta(z).$$

Lemma 2.2. *The function $f : x \mapsto e^{-\pi x^2}$, $x \in \mathbb{R}^n$, is equal to its Fourier transform.*

Proof of Proposition 2.1. The first transformation formula follows directly since θ is a function of q. Since both sides are holomorphic on $\mathcal{H}$, it is sufficient to prove the second formula for $z = it/2$, $t > 0$. We apply the Poisson summation formula to the function $f : x \mapsto e^{-\pi t x^2}$ and the lattice $\mathbb{Z} \subset \mathbb{R}$. Using Lemma 2.2, we have

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{t}} e^{-\pi y^2/t}.$$

Hence,

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t}.$$

This shows the second transformation formula for $z = it/2$. □

We let

$$W_4 := \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix},$$

the matrix of $\mathrm{SL}_2(\mathbb{Z})$ mapping z to $-1/4z$, and note that

Proposition 2.3. *The group* $\langle \Gamma_0(4), W_4 \rangle =: \Gamma_0^+(4)$ *is commensurable with* $\mathrm{SL}_2(\mathbb{Z})$ *and is generated by* T *and* W_4*.*

The two previous propositions and a few additional computations show that θ^4 is a modular form of weight 2 and level $\Gamma_0(4)$ (which is something that we are now familiar with). We can now use the theory of modular forms to easily recover a theorem of Lagrange regarding the decomposition of integers as the sum of four squares. It will in fact come as a corollary of the following result:

Proposition 2.4. *Let* n *be a positive integer. Then the number of representations of* n *as a sum of four squares is 8 times the sum of the positive divisors of* n *which are not multiples of 4.*

Proof. The number $r_4(n)$ of representations of n as a sum of four squares is the coefficient of q^n in the expansion of θ^4. The function θ^4 is a modular form of weight 2 and level $\Gamma_0(4)$. We showed in the last problem sheet that

$$E_2^{(4)}(z) = E_2(z) - 4E_2(4z) = -3 - 24 \sum_{n=1}^{\infty} \tilde{\sigma}_1(n) q^n,$$

where

$$\tilde{\sigma}_1(n) = \sum_{\substack{d|n \\ 4 \nmid d}} d,$$

is also a modular form of $M_2(\Gamma_0(4))$ and since the coefficients of θ^4 and $-\frac{1}{3}E_2^{(4)}$ agree up to order 2, they are equal by Corollary 2.6.8 from the lecture notes. This shows the proposition. □

Corollary 2.5 (Lagrange)**.** *Every positive integer is the sum of four squares.*

3 Even unimodular lattices and modular forms

We now turn our interest to a lattice L of $\mathbb{R}^n$ and assume that it is **even** and **unimodular**. A lattice is said to be even if $\langle x, x \rangle$ is even for any lattice vector x, and it is said to be unimodular if its volume equals 1 (this is defined to be the determinant of a matrix of basis vectors or, equivalently, as $\mathrm{vol}(\mathbb{R}^n/L)$). We note that L is unimodular if and only if $L = L^*$. We will see that this special category of lattices is in fact a source of modular forms. More precisely

Theorem 3.1. *Let* L *be an even unimodular lattice in* $\mathbb{R}^n$*. Then,*

1. *The dimension* n *of* V *is divisible by 8.*

2. *The function* Θ_L *is a modular form of weight* $n/2$*.*

Several intermediate results are needed to prove the theorem. More precisely, we'll need Proposition 1.1, Lemma 2.2 and the following transformation formula for Θ_L:

Lemma 3.2. *We have the identity*

$$\Theta_L\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{n}{2}} \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)} \Theta_{L^*}(z). \tag{1}$$

Proof. Since both sides are holomorphic on $\mathcal{H}$, it suffices to prove that the identity hold for $z = it$, $t > 0$. Lemma 2.2 applied with $x' = \frac{x}{\sqrt{t}}$ gives that the Fourier transform of $e^{-\pi x^2/t}$ is $t^{n/2}e^{-\pi t y^2}$. Therefore we obtain the result directly by applying Poisson's formula to

$$\Theta_L\left(-\frac{1}{it}\right) = \sum_{\ell \in L} e^{\pi i(-1/it)x^2}.$$

□

Proof of Theorem 3.1. Let L be an even unimodular lattice in $\mathbb{R}^n$. Assume for a contradiction that n is not divisible by 8. Replacing L by $L \oplus L$ or by $L \oplus L \oplus L \oplus L$ if necessary, we can assume that $n \equiv 4 \bmod 8$. From Lemma 3.2, we have

$$\Theta_L(S \cdot z) = (-1)^{n/4} z^{n/2} \Theta_L(z) = -z^{n/2}\Theta_L(z).$$

Hence,

$$\Theta_L((TS) \cdot z) = -z^{n/2}\Theta_L(z)$$

and

$$\Theta_L((TS)^3 \cdot z) = -(-1)^{n/2}\Theta_L(z) = -\Theta_L(z).$$

But $(TS)^3 = \mathrm{Id}$, which yields a contradiction. We have proved that n is divisible by 8.

To show that $\Theta_L(z)$ is mpdular of weight $n/2$, simply apply Lemma 3.2. Since $8|n$, we have

$$\Theta_L(S \cdot z) = z^{n/2}\Theta_L(z).$$

This finishes the proof. □

This is the building block at the base of Maryna Viazovska's solution to the sphere packing problem in dimensions 8 and 24, in which she ingeniously uses the theory of modular forms to obtain results on the theta series attached to E_8 lattice and to the Leech lattice.

References

[Bru+08] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder and Don Zagier. **The 1-2-3 of Modular Forms**. Universitext. Springer, Berlin, Heidelberg, 2008.

[Ebe13] Wolfgang Ebeling. **Lattices and Codes**. Advanced Lectures in Mathematics. Springer Spektrum, Wiesbaden, 2013.