Modular Forms: Problem Sheet 10

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1. Let $\Gamma' \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups with $-I \in \Gamma'$. Suppose that $f \in \mathcal{S}_k(\Gamma) \subset \mathcal{S}_k(\Gamma')$ and that $g \in \mathcal{S}_k(\Gamma')$. Letting $\Gamma = \bigcup_i \alpha_i \Gamma'$, define the trace of g to be

$$\operatorname{trace} g = \sum_i g|_k \alpha_i \in \mathcal{S}_k(\Gamma).$$

Show that

$$\langle f, g\rangle_{\Gamma'} = \langle f, \operatorname{trace} g\rangle_{\Gamma}.$$

Solution: By definition, we have

$$\langle f, g\rangle_{\Gamma'} = \int_{D_{\Gamma'}} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau),$$

where $d\mu$ denotes the hyperbolic measure. Check that the coset decomposition $\Gamma = \bigcup_i \alpha_i \Gamma'$ gives a decomposition of $D_{\Gamma'}$ as

$$D_{\Gamma'} = \bigsqcup_i \alpha_i D_{\Gamma}.$$

Hence,

$$\langle f, g\rangle_{\Gamma'} = \sum_i \int_{\alpha_i D_\gamma} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau). \tag{1}$$

Now fix i, and use the invariance of $d\mu$ and the modularity of f to compute

$$\begin{aligned}\int_{\alpha_i D_\Gamma} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) &= \int_{D_\Gamma} f(\alpha_i\tau)\overline{g(\alpha_i\tau)}\Im(\alpha_i\tau)^k d\mu(\alpha_i\tau)\\ &= \int_{D_\Gamma} f|_k\alpha_i(\tau)\overline{g|_k\alpha_i(\tau)}\Im(\tau)^k d\mu(\tau)\\ &= \int_{D_\Gamma} f(\tau)\overline{g|_k\alpha_i(\tau)}\Im(\tau)^k d\mu(\tau).\end{aligned}$$

Using absolute convergence, we can exchange the sum and the integral in (1). This yields the result.

2. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Show that if $\alpha^{-1}\Gamma\alpha = \Gamma$, then $|_k\alpha'$ is the adjoint operator of $|_k\alpha$ with respect to the Petersson inner product.

Solution: Let $f \in \mathcal{S}_k(\Gamma)$, then $f|_k\alpha \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$. Additionally, let $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$. We consider the Petersson inner product of $f|_k\alpha$ and g. To compute it, we'll use the fact that

$D_{\alpha^{-1}\Gamma\alpha} = \alpha D_\Gamma$. Moreover, we notice that $\alpha'\tau = \alpha^{-1}\tau$, for any $\tau \in \mathcal{H}$. We have

$$\begin{aligned}\langle f|_k\alpha, g\rangle_{\alpha^{-1}\Gamma\alpha} &= \int_{D_{\alpha^{-1}\Gamma\alpha}} f|_k\alpha(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau)\\ &= \int_{D_{\alpha^{-1}\Gamma\alpha}} \det(\alpha)^{k-1} f(\alpha\tau)j(\alpha,\tau)^{-k}\overline{g(\tau)}d\mu(\tau)\\ &= \int_{D_\Gamma} \det(\alpha)^{k-1} f(\tau)j(\alpha,\alpha'\tau)^{-k}\overline{g(\alpha'\tau)}\Im(\alpha'\tau)^k d\mu(\alpha'\tau)\end{aligned}$$

Now recall that $\det(\alpha) = j(\alpha\alpha',\tau) = j(\alpha,\alpha'\tau)j(\alpha',\tau)$ and that $d\mu$ is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$. Additionally, compute that

$$\Im(\alpha'\tau) = \det(\alpha')|j(\alpha',\tau)|^{-2}\Im(\tau) = \det(\alpha)|j(\alpha',\tau)|^{-2}\Im(\tau).$$

The above equation then becomes

$$\langle f|_k\alpha, g\rangle_{\alpha^{-1}\Gamma\alpha} = \int_{D_\Gamma} f(\tau)\overline{g|_k\alpha'(\tau)}\Im(\tau)^k d\mu(\tau) = \langle f, g|_k\alpha'\rangle_\Gamma.$$

Since we assumed that $\Gamma = \alpha^{-1}\Gamma\alpha$, this proves the claim.

3. Define the normalized Petersson inner product as

$$[f,g]_\Gamma = \frac{1}{V_\Gamma}\langle f,g\rangle_\Gamma,$$

for any two cusp forms $f, g \in \mathcal{S}_k(\Gamma)$, and with $V_\Gamma = \int_{D_\Gamma}\frac{dxdy}{y^2}$.

i. Find a formula relating the volumes V_Γ and $V_{\mathrm{SL}_2(\mathbb{Z})}$ and the index $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$, where $\overline{\Gamma} = \{\pm \mathrm{Id}\}\Gamma/\{\pm \mathrm{Id}\}$.

Solution: The formula in question is

$$V_\Gamma = [\mathrm{SL}_2(\mathbb{Z}) : \{\pm \mathrm{Id}\}\Gamma]V_{\mathrm{SL}_2(\mathbb{Z})}.$$

Since Γ is a congruence subgroup, $\{\pm \mathrm{Id}\}\Gamma$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. Hence there exist a set $\{\alpha_i\} \subset \mathrm{SL}_2(\mathbb{Z})$ of cardinality $[\mathrm{SL}_2(\mathbb{Z}) : \{\pm \mathrm{Id}\}\Gamma]$ such that

$$D_\Gamma = \bigsqcup_i \alpha_i D_{\mathrm{SL}_2(\mathbb{Z})}.$$

Hence

$$\int_{D_\Gamma} d\mu(\tau) = \sum_i \int_{\alpha_i D_{\mathrm{SL}_2(\mathbb{Z})}} d\mu(\tau) = \sum_i \int_{D_{\mathrm{SL}_2(\mathbb{Z})}} d\mu(\alpha_i\tau).$$

Since the hyperbolic measure is $\mathrm{SL}_2(\mathbb{R})$-invariant, the previous sum does not depend on the choice of coset representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash\{\pm \mathrm{Id}\}\Gamma$ and each integral equals $V_{\mathrm{SL}_2(\mathbb{Z})}$. This shows the statement.

ii. Show that if $\Gamma' \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ are congruence subgroups, then $[,]_\Gamma = [,]_{\Gamma'}$ on $\mathcal{S}_k(\Gamma)$.

Solution: We start by noting that if

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \alpha_i \{\pm \mathrm{Id}\}\Gamma$$
$$\{\pm \mathrm{Id}\}\Gamma = \bigsqcup_j \beta_j \{\pm \mathrm{Id}\}\Gamma'$$

then

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{i,j} \alpha_i \beta_j \Gamma'.$$

Therefore, using the equation from one for Γ and Γ', we obtain

$$V_{\Gamma'} = [\overline{\Gamma} : \overline{\Gamma'}] V_\Gamma.$$

Now let $f, g \in \mathcal{S}_k(\Gamma)$. We have

$$\begin{aligned} [f,g]_{\Gamma'} &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}] V_\Gamma} \int_{D_{\Gamma'}} f\overline{g}\Im(\tau)^k d\mu(\tau) \\ &= \frac{1}{[\overline{\Gamma} : \overline{\Gamma'}] V_\Gamma} \sum_j \int_{\beta_j D_\Gamma} f\overline{g}\Im(\tau)^k d\mu(\tau) \end{aligned}$$

Considering the last integral for an arbitrary j, we use the Γ'-invariance of the integral and obtain

$$\int_{\beta_j D_\Gamma} f\overline{g}\Im(\tau) d\mu(\tau) = \int_{D_\Gamma} f\overline{g}\Im(\tau)^k d\mu(\tau).$$

Since $|\{\beta_j\}| = [\overline{\Gamma} : \overline{\Gamma'}]$ we get the result.