# Modular Forms: Problem Sheet 11

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1. (a) Let $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Show that $\gamma^{-1}\Gamma_1(N)\gamma = \Gamma_1(N)$, and so the operator $w_N = |_k\gamma$ is the double coset operator $[\Gamma_1(N)\gamma\Gamma_1(N)]_k$ on $\mathcal{S}_k(\Gamma_1(N))$. Show that $w_N \langle n \rangle w_N^{-1} = \langle n \rangle^*$ for all $n$ such that $(n, N) = 1$, and thus for all $n$.

**Solution:** We first note that, for $a, b, c, d \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$\gamma^{-1} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \gamma = \begin{pmatrix} d & -c \\ -Nb & a \end{pmatrix} \in \Gamma_1(N).$$

Hence we have $\Gamma_1(N) = \gamma^{-1}\Gamma_1(N)\gamma$. Additionally, we have the following coset decomposition:

$$\Gamma_1(N)\gamma\Gamma_1(N) = \gamma\Gamma_1(N) = \Gamma_1(N)\gamma.$$

This shows $w_N := |_k\gamma = [\Gamma_1(N)\gamma\Gamma_1(N)]_k$. Similarly, we see that

$$\gamma\Gamma_1(N)\gamma^{-1} = \Gamma_1(N),$$

and therefore that $w_N^{-1} = |_k\gamma^{-1} = [\Gamma_1(N)\gamma^{-1}\Gamma_1(N)]_k$.
Recall that for $n \in \mathbb{Z}$ with $(n, N) = 1$ we computed

$$\langle n \rangle^* = \langle n^{-1} \rangle .$$

On the other hand, for $\left(\begin{smallmatrix} a & b \\ c & n \end{smallmatrix}\right) \in \Gamma_0(N)$,

$$\begin{aligned} w_N \langle n \rangle w_N^{-1} &= \left[\Gamma_1(N)\gamma \begin{pmatrix} a & b \\ c & n \end{pmatrix} \gamma^{-1}\Gamma_1(N)\right]_k \\ &= \left[\Gamma_1(N) \begin{pmatrix} n & -c/N \\ -Nb & a \end{pmatrix} \Gamma_1(N)\right]_k . \end{aligned}$$

We clearly have $\left(\begin{smallmatrix} n & -c/N \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_0(N)$ and $an \equiv 1 \mod N$. Hence,

$$w_N \langle n \rangle w_N^{-1} = \langle n^{-1} \rangle = \langle n \rangle^* .$$

In the case $(n, N) > 1$, recall that then the diamond operator $\langle n \rangle$ is the null operator.

(b) Let $\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N) = \cup_j \Gamma_1(N)\beta_j$ be a distinct union. From part (a), $\gamma\Gamma_1(N) = \Gamma_1(N)\gamma$ and similarly for $\gamma^{-1}$. Use this and the formula

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \gamma^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \gamma$$

to find coset representatives for $\Gamma_1(N)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_1(N)$. Use the formula for the adjoint computed in Theorem 3.4.6 and the coset representatives to show that

$$T_p^* = w_N T_p w_N^{-1} \text{ and so } T_n^* = w_N T_n w_N^{-1} \text{ for all } n.$$

**Solution:** A straight-forward computation using the given formula shows that

$$\Gamma_1(N)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\Gamma_1(N) = \Gamma_1(N)\gamma^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\gamma\Gamma_1(N)$$
$$= \bigcup_j \Gamma_1(N)(\gamma^{-1}\beta_j\gamma). \tag{1}$$

Therefore the set of $\delta_j := \gamma^{-1}\beta_j\gamma$ is a set of coset representatives for $\Gamma_1(N)\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right)\Gamma_1(N)$. Recall that $T_p$ was defined as the double coset operator $[\Gamma_1(N)\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)\Gamma_1(N)]_k$. Using the formula from Theorem 3.4.6., we have

$$T_p^* = \left[\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}'\Gamma_1(N)\right]_k = \left[\Gamma_1(N)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\Gamma_1(N)\right]_k,$$

where $\alpha' = \det(\alpha)\alpha^{-1}$ for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Using the coset decomposition (1), we conclude that

$$\left[\Gamma_1(N)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\Gamma_1(N)\right]_k = \sum_j \cdot|_k(\gamma^{-1}\beta_j\gamma) = w_N T_p w_N^{-1}.$$

Note that this computation of the adjoint does not depend on a coprimality condition between $p$ and $N$, and therefore the generalization to $T_n$ for any $n \in \mathbb{Z}_{>0}$ is direct from the definitions (multiplicativity for products of distinct primes and induction for prime powers).

(c) Show that $w_N^* = (-1)^k w_N$ and that $i^k w_N T_n$ is self-adjoint.

2. Let $l$ be a prime and let $N \in \mathbb{Z}_{\geq 1}$. Recall the $U_l$ and $V_l$ operators and their action on the $q$-expansion of $f = \sum_n a_n q^n \in \mathcal{M}_k(\Gamma_1(N))$:

$$U_l f = \sum_n a_{nl} q^n,$$
$$V_l f = \sum_n a_n q^{nl}.$$

Compute the composition $V_l \circ U_l$.

**Solution:** By looking at the action on $q$-expansions, we see that $V_l \circ U_l$ discards all coefficients of $f$ except those corresponding to a power of $q$ that is divisible by $p$.

3. Recall that a lattice in $\mathbb{C}$ is a subgroup $\Lambda \subset \mathbb{C}$ of the form

$$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

for two $\mathbb{R}$-linearly independent complex numbers $\omega_1, \omega_2$. We make the normalizing convention that $\omega_1/\omega_2 \in \mathcal{H}$ and denote $\mathcal{L}$ the set of such lattices. Note that two lattices of $\mathcal{L}$, say $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $\Lambda' = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$, are equal if and only if there exists a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \gamma\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}.$$

For an arbitrary $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, we let $\tau = \omega_1/\omega_2 \in \mathcal{H}$ and $\Lambda_\tau := \tau\mathbb{Z} + \mathbb{Z}$. Then, the map

$$\varphi_\tau : \begin{array}{ccc}\mathbb{C}/\Lambda & \to & \mathbb{C}/\Lambda_\tau\\ z + \Lambda & \mapsto & z/\omega_2 + \Lambda_\tau\end{array}$$

is an isomorphism. Moreover, $\tau \in \mathcal{H}$ is uniquely determined up to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

Finally, we say that a map

$$\mathcal{F} : \mathcal{L} \to \mathbb{C}$$

is homogeneous of weight $k \in \mathbb{Z}$ if it satisfies

$$\mathcal{F}(\lambda\Lambda) = \lambda^{-k}\mathcal{F}(\Lambda) \text{ for all } \lambda \in \mathbb{C}^* \text{ and } \Lambda \in \mathcal{L}.$$

Note that the value of $F$ at any $\Lambda \in \mathcal{L}$ is completely determined by its value at $\Lambda_\tau$. To a homogeneous map, we associate a function $f : \mathcal{H} \to \mathbb{C}$ by letting $f(\tau) = \mathcal{F}(\Lambda_\tau)$ for all $\Lambda \in \mathcal{L}$. This function is then well-defined by the above paragraph.

(a) Show that such an $f$ is modular, i.e. prove that $f(\gamma\tau) = \mathcal{F}(\Lambda_{\gamma\tau}) = j(\gamma, \tau)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

**Solution:** This is a straight-forward computation. Letting $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned} f(\gamma\tau) = \mathcal{F}(\Lambda_{\gamma\tau}) &= \mathcal{F}\left(\frac{a\tau+b}{c\tau+d}\mathbb{Z} + \mathbb{Z}\right) \\ &= \mathcal{F}\left(\frac{1}{c\tau+d}[(a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z}]\right) \\ &= (c\tau+d)^k \mathcal{F}((a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z}). \end{aligned}$$

Now, the lattice $(a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}$ is equal to $\Lambda_\tau$ since the basis vectors $a\tau + b$ and $c\tau + d$ are related to the basis vectors $\tau, 1$ of $\Lambda_\tau$ by multiplication by an element of $\mathrm{SL}_2(\mathbb{Z})$. We conclude that

$$f(\gamma\tau) = \mathcal{F}(\Lambda_{\gamma\tau}) = (c\tau+d)^k\mathcal{F}(\Lambda_\tau) = (c\tau+d)^k f(\tau).$$

(b) For a prime number $p$, define the Hecke operator $T_p$ by letting $T_p\mathcal{F}$ be the homogeneous function of weight $k$ corresponding to $T_pf$ (where the second occurence of $T_p$ is our usual Hecke operator on $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$). Show that

$$T_p\mathcal{F}(\Lambda) = \frac{1}{p}\sum_{\substack{\Lambda' \supset \Lambda \\ [\Lambda':\Lambda]=p}} \mathcal{F}(\Lambda'), \ \forall \Lambda \in \mathcal{L}.$$

**Solution:** By homogeneity of $\mathcal{F}$, it is enough to show the statement for $\Lambda = \Lambda_\tau$, $\tau \in \mathcal{H}$. Since we are working at level $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$, any diamond operator is the identity and we do not have to consider the case of a prime $p$ dividing the level. Hence,

$$\begin{aligned} T_p\mathcal{F}(\Lambda_\tau) &= T_pf(\tau) \\ &= p^{k-1}p^{-k}\sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right) + p^{k-1}f(p\tau) \\ &= \frac{1}{p}\sum_{j=0}^{p-1}\mathcal{F}\left(\frac{\tau+j}{p}\mathbb{Z} + \mathbb{Z}\right) + p^{k-1}\mathcal{F}(p\tau\mathbb{Z} + \mathbb{Z}) \\ &= \frac{1}{p}\left[\sum_{j=0}^{p-1}\mathcal{F}\left(\frac{\tau+j}{p}\mathbb{Z} + \mathbb{Z}\right) + \mathcal{F}\left(\tau\mathbb{Z} + \frac{1}{p}\mathbb{Z}\right)\right]. \end{aligned}$$

To conclude, we have to show that the lattices $\frac{\tau+j}{p}\mathbb{Z} + \mathbb{Z}$ for $0 \le j \le p-1$ and $\tau\mathbb{Z} + \frac{1}{p}\mathbb{Z}$ are precisely the lattices in which $\Lambda_\tau$ has index $p$.

We proceed by first letting $M_p$ be the set of all lower upper triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ such that $ad = p$ and $0 \leq b < d$, i.e.

$$M_p = \left\{ \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \mid 0 \leq j < p \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and then showing that the map

$$\begin{array}{ccc} M_p & \to & \{\Lambda' \in \mathcal{L} \mid [\Lambda' : \Lambda_\tau] = p\} \\ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) & \mapsto & \frac{1}{p}((a\tau + b)\mathbb{Z} + d\mathbb{Z}) \end{array}$$

is a bijection. Indeed, note that any lattice $\Lambda'$ containing $\Lambda_\tau$ as a sublattice of index $p$ admits basis vectors $\omega_1, \omega_2$ of the form

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{1}{p} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an integral matrix of determinant $p$. We know from linear algebra that an integral matrix of determinant $p$ can be transformed to a unique element of $M_p$ using row transformations. This shows that any such $\Lambda'$ admits a basis of the form $\frac{1}{p}(a\tau + b)$, $\frac{1}{p}d$ for some $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in M_p$.

**Remark 1.** The same statement and proof apply replacing $\Lambda_\tau$ by an arbitrary $\Lambda$.

**Remark 2.** The $T_p$ Hecke operator, viewed as an operator on homogeneous lattice functions, acts by "averaging" the function on the "sup-lattices" containing the given lattice with index $p$. (It is not exactly an average since there are $p + 1$ such lattices.)