Modular Forms: Problem Sheet 1

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1. Show that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$ is **transitive** (for any $z, z' \in \mathcal{H}$, there is $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g \circ z = z'$) and **faithful** (no non-identity element in $\mathrm{PSL}_2(\mathbb{R})$ acts trivially on $\mathcal{H}$).

   Answer. We show that for any $z = x + iy \in \mathcal{H}$, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ so that $g \circ i = z$. We have,
$$g \circ i = \frac{ai + b}{ci + d} = \frac{ac + bd}{c^2 + d^2} + \frac{i}{c^2 + d^2}.$$
   Setting $x + iy = g \circ i$ and solving for the entries of g we find
$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \circ i = z.$$
   We conclude that for any $z \in \mathcal{H}$, $z \in \mathrm{PSL}_2(\mathbb{R}) \circ i$ and therefore that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$ is transitive.

   We now show that the action is faithful. Assume that $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that $\gamma \circ z = z$ for all $z \in \mathcal{H}$. Since $\Im(\gamma \circ z) = \frac{\Im(z)}{|cz+d|^2}$, we must have $|cz + d|^2 = 1$ for all $z \in \mathcal{H}$. We first consider the case $c \neq 0$ and set $z = i/c$. Then $d = 0$ must hold and equating the real parts of $\gamma \circ z$ and z we see that $a = 0$ must also hold. We conclude that, to fix i/c, we must have $\gamma = \left(\begin{smallmatrix} 0 & -1/c \\ c & 0 \end{smallmatrix}\right)$. However, this obviously doesn't fix the whole of $\mathcal{H}$. We have shown that c must vanish. The equation $|cz + d|^2 = 1$ then implies $d = \pm 1$. Let us set $d = 1$ (the other case is done similarly). Again equating the real parts of $\gamma \circ z$ and z, we see that, to fix $z = x + iy$ the equation
$$b + ax = x$$
   has to hold for all $x \in \mathbb{R}$. Of course, this can only hold if $b = 0$ and $a = 1$, which yields $\gamma = \mathrm{Id}$. This concludes the proof. ◀

2. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbb{Z})$. Show that if $c = 1$ and $d = 0$, and there is some $z \in D$ such that $\gamma z \in D$, then we must have one of the following possibilities:

   - $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and z is any element of D with $|z| = 1$,
   - $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $z = \gamma z = 1 + \rho$,
   - $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $z = \gamma z = \rho$.

   (This fills in a detail of Step 2 of Theorem 1.2.2.)

   Answer. Recall that in the proof of Step 2 of Theorem 1.2.2. we showed that in the case $c = 1$, $d = 0$, if there is some $z \in D$ such that $\gamma \circ z \in D$, then $|z| = 1$. Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\det(\gamma) = 1$ so we must also have $b = -1$. Letting $z = x + iy$, we now compute
$$\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \circ z = a - x + iy.$$

From this we deduce two things. First, that $\Im(\gamma \circ z) = \Im(z)$. Second, since $z, \gamma \circ z \in D$, we have

$$x + \tfrac{1}{2} \geq a \geq x - \tfrac{1}{2} \quad \text{and} \quad \tfrac{1}{2} \geq x \geq -\tfrac{1}{2}.$$

We deduce that the only possible values for $a \in \mathbb{Z}$ are 1, -1 and 0. Moreover, the case $a = 1$ can only occur if $x = 1/2$, hence if $z = \rho + 1$. The case $a = -1$ can only occur if $x = -1/2$, hence if $z = \rho$. Finally, for $a = 0$, the action of γ just sends z to its reflection with respect to the imaginary axis, hence $\gamma \circ z \in D$ for any $z \in D$ with $|z| = 1$. ◀

3. Express the element

$$\gamma = \begin{pmatrix} 8 & 29 \\ 11 & 40 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

in terms of the generators S and T.

Answer. We give a general algorithm to help us solve such problems. First note that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad T^n \begin{pmatrix} a & b \\ c & b \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}.$$

Let. $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Assume moreover that $c \neq 0$. If $|a| \geq |c|$, start by dividing a by c (otherwise exchange the two rows using S, then proceed as follows). More precisely, if $a = qc + r$ for some $r \in \mathbb{Z}$ with $|r| < |c|$, then the upper left entry of $T^{-q}g$ is r. If $r \neq 0$, we then exchange the two rows using S and repeat the process with $-c$ in the role of a and r in the role of c. Continuing this way as long as the lower left entry isn't zero, we will arrive in a finite number of steps to a matrix of $\mathrm{SL}_2(\mathbb{Z})$ whose lower left entry vanishes. Since the resulting matrix still has determinant 1, it must be of the form

$$\begin{pmatrix} \mathrm{sgn} \cdot 1 & \beta \\ 0 & \mathrm{sgn} \cdot 1 \end{pmatrix}, \ \mathrm{sgn} \in \{\pm 1\}.$$

This is either equal to T^{β} or to $-T^{-\beta}$ for some $m \in \mathbb{Z}$. Hence, there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma \cdot g = \mathrm{sgn}\, T^{\mathrm{sgn}\,\beta}.$$

The right-hand side of the previous equation can be expressed in terms of S and T since $T^{\mathrm{sgn}\,\beta} \in \mathrm{SL}_2(\mathbb{Z})$ and $-\mathrm{Id} = S^2 \in \mathrm{SL}_2(\mathbb{Z})$. We conclude that the expression of g in terms of S and T is

$$\gamma^{-1} S^2 T^{-\beta}, \text{ if } \mathrm{sgn} = -1 \quad \text{and} \quad \gamma^{-1} T^{\beta}, \text{ if } \mathrm{sgn} = 1.$$

Applying the algorithm in the context of this exercise, we find

$$\gamma = S^3 T^{-1} S^3 T^2 S^3 T^{-1} S^3 T^2 S T^4.$$

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4. Let $G = \mathrm{SL}_2(\mathbb{Z})$. Show that for $z \in D$, we have

$$G_z = \begin{cases} C_6 = \langle ST \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle & z = \rho \\ C_6 = \langle TS \rangle = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle & z = \rho + 1 \\ C_4 = \langle S \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle & z = i \\ C_2 = \langle -\mathrm{Id} \rangle = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle & z \notin \{i, \rho, \rho + 1\} \end{cases}$$

Answer. In Step 2 of the proof of Theorem 1.2.2, we have determined for which $z \in D$ there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ z \in D$ and we have described every such γ. Of course, for any $z \in D$ we have

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \circ z = z\} \subset \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \circ z \in D\}.$$

So, to solve this problem, compute for each case listed in Step 2 which γ's fix z, and observe that the ones that do belong to the cyclic groups presented above. ◀