Modular Forms: Problem Sheet 2

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1. (a) Let a, b, c, d be integers. Show that the map $(m, n) \mapsto (ma + nc, mb + nd)$ is a bijection on $\mathbb{Z}^2 - \{(0, 0)\}$ if and only if $ad - bc = \pm 1$.

Solution: It is clear that

$$(m, n) \mapsto (ma + nc, mb + nd)$$

is a well-defined map from $\mathbb{Z}^2$ to itself. Let A be the matrix of $M_2(\mathbb{Z})$ associated to this linear transformation.
Assume first that $\det(A) \in \{\pm 1\}$. Then the inverse A^{-1} belongs to $M_2(\mathbb{Z})$. Finally,

$$A(A^{-1}v) = v = A^{-1}(Av)$$

shows that the map is bijective (existence of a left- and right-inverse).
Assume now that the map is bijective. It can be viewed as a linear map on the $\mathbb{R}$-vector space $\mathbb{R}^2$ that maps the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ to itself bijectively. By definition, the volume of the fundamental parallelogram of $A\mathbb{Z}^2$ equals $|\det(A)|$. Since $A\mathbb{Z}^2 = \mathbb{Z}^2$,

$$|\det(A)| = \mathrm{vol}(\mathbb{R}^2/A\mathbb{Z}^2) = \mathrm{vol}(\mathbb{R}^2/\mathbb{Z}^2) = 1$$

and we have proved the statement.

(b) Hence show that for $k \geq 4$ even, the Eisenstein series

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k}$$

satisfies $G_k|_k\gamma = G_k$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, without using Theorem 1.2.2 (a) as we did in class. (You may assume that the sum defining $G_k(z)$ converges absolutely for any $z \in \mathcal{H}$.)

Solution: Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\begin{aligned}
G \mid_k \gamma(z) &= \sum_{(m,n) \neq (0,0)} \frac{(cz + d)^{-k}}{(m(\gamma \circ z) + d)^k} \\
&= \sum_{(m,n) \neq (0,0)} \frac{1}{(m(az + b) + n(cz + d))^k} \\
&= \sum_{(m,n) \neq (0,0)} \frac{1}{((ma + nc)z + (mb + nd))^k} \\
&= G_k(z),
\end{aligned}$$

where the last equality follows from $\det(\gamma) \in \{\pm 1\}$ by (a).

2. (a) Prove the identity

$$\pi z \frac{\cos(\pi z)}{\sin(\pi z)} = \sum_{k \geq 0,\, \text{even}} (2\pi i)^k \frac{B_k}{k!} z^k \qquad \forall\, |z| < 1.$$

Solution: We have

$$\begin{aligned}\pi z\cot(\pi z) =& i\pi z\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = i\pi z\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1}\\ =& i\pi z\left(1+\frac{2}{e^{2i\pi z}-1}\right) = i\pi z + \frac{2i\pi z}{e^{2i\pi z}-1}\\ =& i\pi z + \sum_{k=0}^{\infty}\frac{B_k}{k!}(2\pi i z)^k\\ =& \sum_{k\geq 0,\,\text{even}}(2\pi i)^k\frac{B_k}{k!}z^k,\end{aligned}$$

where the last equality follows from (d). Note that convergence holds on $|z|<1$.

(b) Show that

$$\pi z\frac{\cos(\pi z)}{\sin(\pi z)} = 1 - 2\sum_{k\geq 2,\,\text{even}}\zeta(k)z^k \qquad \forall\, |z|<1.$$

Solution: Starting from the equation

$$\sum_{d\in\mathbb{Z}}\frac{1}{z+d} = \pi\cot(\pi z)$$

seen in the lecture, we get

$$\begin{aligned}\pi z\cot(\pi z) =& z\left[\frac{1}{z}+\sum_{d=1}^{\infty}\left(\frac{1}{z+d}+\frac{1}{z-d}\right)\right]\\ =& 1+2z^2\sum_{d=1}^{\infty}\frac{1}{z^2-d^2} = 1+2z^2\sum_{d=1}^{\infty}\frac{1}{d^2(z^2/d^2-1)}\\ =& 1-2z^2\sum_{d=1}^{\infty}\sum_{k=0}^{\infty}\frac{z^{2k}}{d^{2k+2}}\\ =& 1-2\sum_{k=1}^{\infty}\zeta(2k)z^{2k},\end{aligned}$$

where absolute convergence is guaranteed on $|z|<1$ (additionally recall that the Riemann zeta $\zeta(s)$ converges absolutely on $\Re(s)>1$).

(c) Deduce that the values of the Riemann zeta function at the even integers $k\geq 2$ are given by

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2\cdot k!}$$

Solution: This follows directly from the uniqueness of power series on their domain of convergence. More precisely,

$$\sum_{k\geq 0,\,\text{even}}(2\pi i)^k\frac{B_k}{k!}z^k = 1-2\sum_{k\geq 2,\,\text{even}}^{\infty}\zeta(k)z^k$$

on $|z|<1$, hence their coefficients must coincide for each $k\geq 0$.

(d) Prove that B_k is non-zero if and only if k is 1 or k is even.

Solution: Consider

$$\frac{t}{e^t - 1} + \frac{-t}{e^{-t} - 1} = 2\frac{t}{e^t - 1} + t.$$

Now, the left-hand side equals

$$\sum_{k \geq 0} \frac{B_k}{k!} t^k + \sum_{k \geq 0} \frac{B_k}{k!} (-t)^k = 2 \sum_{k \geq 0,\, \text{even}} \frac{B_k}{k!} t^k$$

and the right-hand side equals

$$2 \sum_{k \geq 0} \frac{B_k}{k!} t^k + t.$$

Hence, by uniqueness of the power series on their domain of convergence we conclude that $B_1 = -\frac{1}{2}$ and that B_k vanishes for all odd $k \geq 3$.

3. (a) Show that the Eisenstein series E_4 has a simple zero at $z = \rho$ and no other zeros.

Solution: Lemma 1.4.3. from the lecture notes shows that E_4 is holomorphic on the whole of $\mathcal{H}$, hence that $v_P(E_4) \in \mathbb{Z}_{\geq 0}$ for any representative P of a $\mathrm{PSL}_2(\mathbb{Z})$ orbit in $\mathcal{H}$. Moreover, Proposition 1.4.5. shows that it is holomorphic at ∞ with $v_\infty(E_4) = 0$. At the same time, the valence formula states that

$$v_\infty(E_4) + \frac{v_\rho(E_4)}{3} + \frac{v_i(E_4)}{2} + \sum_{P \in W} v_P(E_4) = \frac{1}{3},$$

where W contains a unique representative for every $\mathrm{PSL}_2(\mathbb{Z})$-orbit passing through $\mathcal{H}$, apart from the orbits of i and ρ. This implies

$$\frac{v_i(E_4)}{2} + \sum_{P \in W} v_P(E_4) = \frac{1 - v_\rho(E_4)}{3}.$$

Now notice that the left-hand side above is in $\mathbb{Z}_{\geq 0} + \frac{1}{2}\mathbb{Z}_{\geq 0}$, so in particular it is positive, while the right-hand side is in $\frac{1}{3}(1 - \mathbb{Z}_{\geq 0})$ and will be positive iff $v_\rho(E_4) \in \{0, 1\}$. For this equation to hold, we must then have $v_i(E_4) = 0 = v_P(E_4)$ for all $P \in W$ and $v_\rho(E_4) = 1$.

(b) Similarly, show that E_6 has a simple zero at $z = i$ and no other zeros.

Solution: This is solved similarly as the part (a).