Modular Forms: Problem Sheet 8

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1. Find an integer N and a Dirichlet character χ modulo N such that

   (a) χ is primitive, but not injective;

   > **Solution:** Let $N = 8$, then $(\mathbb{Z}/8\mathbb{Z})^* \cong C_2 \times C_2$, and define the Dirichlet character
   >
   > $$\chi : \begin{array}{ccr} 1 & \mapsto & 1 \\ 3 & \mapsto & 1 \\ 5 & \mapsto & -1 \\ 7 & \mapsto & -1 \end{array}$$
   >
   > This character is clearly not injective. Is it primitive? The positive divisors of 8 are 2 and 4. Modulo 2, the only Dirichlet character is the trivial one. Modulo 4, we have observed that there exists a unique non-trivial Dirichlet character χ_4 mapping 3 to -1. Its induced character mod 8 is by definition $\tilde{\chi}(a) = \chi_4(a \bmod 4)$, and it is definitely different from χ.

   (b) χ is injective, but not primitive.

   > **Solution:** Let $N = 6$, then $(\mathbb{Z}/6\mathbb{Z})^* \cong C_2$, and define
   >
   > $$\chi : \begin{array}{ccr} 1 & \mapsto & 1 \\ 5 & \mapsto & -1 \end{array}$$
   >
   > Note that χ is clearly injective. However, it is an easy check to see that χ is induced by the character mod 3 mapping 2 to -1.

   Show that the situation of (b) can only occur if $N = 2 \bmod 4$.

> **Solution:** We let $N = 2^\alpha p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, where $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and where the p_i's are pairwise distinct odd primes. If χ is a non-primitive Dirichlet character mod N, then χ can be written as $\chi = \tilde{\chi} \circ \pi$, where $\tilde{\chi}$ is a character mod M for a positive divisor M of N and $\pi : (\mathbb{Z}/N\mathbb{Z})^* \to (\mathbb{Z}/M\mathbb{Z})^*$ is the reduction mod M. Note that χ is injective if and only if both $\tilde{\chi}$ and π are injective. We show that for $\alpha = 0$ and $\alpha \geq 2$, π cannot be injective.
>
> We write $M = 2^\beta p_1^{\beta_1} \cdots p_n^{\beta_n}$, where $\beta, \beta_i \in \mathbb{N}$, $\beta \leq \alpha$, and $\beta_i \leq \alpha_i$ for all i such that if $\beta = \alpha$ then $\beta_i < \alpha_i$ for some i. We then have the following diagram:
>
> $$\begin{tikzcd} (\mathbb{Z}/N\mathbb{Z})^* \arrow[r, "\cong"] \arrow[d, "\pi"] & (\mathbb{Z}/2^\alpha\mathbb{Z})^* \times (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_n^{\alpha_n}\mathbb{Z})^* \arrow[ddl, dashed, "\tilde{\pi}"] \\ (\mathbb{Z}/M\mathbb{Z})^* \arrow[d, "\cong"] & \\ (\mathbb{Z}/2^\beta\mathbb{Z})^* \times (\mathbb{Z}/p_1^{\beta_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_n^{\beta_n}\mathbb{Z})^* & \end{tikzcd}$$

where $\tilde{\pi}(u, v_1, \ldots, v_n) = (u \bmod 2^\beta, v_1 \bmod p_1^{\beta_1}, \ldots, v_n \bmod p_n^{\beta_n})$ is simply defined as composition of the three other morphisms. Recalling that

$$(\mathbb{Z}/2^\alpha\mathbb{Z})^* \cong \begin{cases} C_1, & \alpha = 1 \\ C_2, & \alpha = 2 \\ C_2 \times C_{2^{\alpha-2}}, & \alpha > 2 \end{cases}$$

and that $(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}) \cong C_{\varphi(p_i^{\alpha_i})}$, where φ denotes the Euler totient function. We can now deduce when $\tilde{\pi}$ is injective in function of α by conducting a simple case by case analysis.

2. Let p and q be distinct primes dividing the positive integer N. Show directly that if $\alpha_0, \ldots, \alpha_{p-1}$ and $\beta_0, \ldots, \beta_{q-1}$ are the left coset representatives for the double cosets T_p and T_q constructed in Proposition 3.2.2, then $\{\alpha_i\beta_j : 0 \le i \le p-1, 0 \le j \le q-1\}$ is a set of left coset representatives for the double coset $\Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix}\Gamma_1(N)$.

   Hence give a direct proof that $T_pT_q = T_qT_p$ in this case.

Solution: We have shown that we can take

$$\alpha_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \ \beta_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}.$$

We compute that

$$\alpha_i\beta_j = \begin{pmatrix} 1 & j+iq \\ 0 & pq \end{pmatrix}.$$

Now, note that

$$\{j + iq \mid 0 \le j \le q-1,\ 0 \le i \le p-1\} = \{k \mid 0 \le k \le pq-1\}$$

and define

$$\gamma_k := \begin{pmatrix} 1 & k \\ 0 & pq \end{pmatrix}.$$

Showing that $\{\gamma_k\}$ is a set of left coset representatives for the double coset $\Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix}\Gamma_1(N)$ can be done using a similar method as the one presented in the lecture and in the exercise class. As a byproduct, we obtain the commutativity of T_p and T_q since $\{\alpha_i\beta_j\} = \{\beta_j\alpha_i\}$.

3. Suppose p is a prime, $\Gamma = \Gamma_1(p)$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Find p matrices $(g_j)_{j=0,\ldots,p-1}$ in $\mathrm{GL}_2^+(\mathbf{Q})$ such that

$$\Gamma g \Gamma = \bigsqcup_{0 \le j < p} \Gamma g_j = \bigsqcup_{0 \le j < p} g_j \Gamma.$$

Solution: Many thanks to Haoran Liang for his input to the following solution. We first prove a lemma that gives an explicit way to construct simultaneous left and right coset representatives. This will give us a recipe to obtain the desired representatives, but you could also have solved the exercise without this result by making a few good guesses.

Lemma 0.1. *Let G be a group and H a subgroup of G of finite index m. Then there exist $g_1, g_2, \ldots, g_m \in G$ such that the g_i's are simultaneously left and right coset representatives.*

Proof. Let $R \subset G$ be a set of double coset representatives so that $G = \bigsqcup_{g \in R} HgH$. Assume that we also have a left and right coset decomposition of HgH as

$$\bigsqcup_{1 \le i \le t} Hga_i = HgH = \bigsqcup_{1 \le i \le t} b_i gH,$$

where t is the index of $H \cap g^{-1}Hg$ in H (which equals the index of $gHg^{-1} \cap H$ in H) and with $a_i, b_i \in H$ for all i. This implies that

$$\bigsqcup_{1 \le i \le t} Hb_i ga_i = HgH = \bigsqcup_{1 \le i \le t} b_i ga_i H. \tag{1}$$

Hence $\{b_i ga_i \mid 1 \le i \le t\}$ is a set of simultaneous left and right double coset representatives for HgH and

$$\bigcup_{g \in R} \{b_i(g) g a_i(g) \mid 1 \le i \le t(g)\},$$

where $a_i(g)$, $b_i(g)$, $t(g)$ are the a_i, b_i, t corresponding to a given g as above, is the set of g_j's mentioned in the statement of the lemma. $\square$

Let us return to our original problem. We will not need the part of the lemma giving a decomposition of G, but we will use (1). We compute that

$$\begin{aligned} &g^{-1}\Gamma g \cap \Gamma = \Gamma(p), \text{ the principal congruence subgroup of level } p \text{ and} \\ &g\Gamma g^{-1} \cap \Gamma = g\Gamma(p)g^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p) \mid p^2 | c \right\} =: \Gamma'. \end{aligned}$$

You can easily check, using the method seen in the lecture, that a set of right coset representatives of Γ/Γ' is given by

$$\left\{ \beta_i := \begin{pmatrix} 1 & 0 \\ pi & 1 \end{pmatrix} \mid 0 \le i \le p-1 \right\}.$$

Easy exercise: Adapting the proof of Proposition 3.1.4., prove that given $\{\beta_i\}$, the set $\{\beta_i g\}$ is a set of right coset representatives for the double coset $\Gamma g \Gamma$.

By (1) in the proof of the above lemma, we conclude that

$$\{\beta_i g \alpha_i \mid 0 \le i \le p-1\}$$

is a set of simultaneous left and right coset representatives, i.e. that

$$\bigsqcup_{0 \le i \le p-1} H\beta_i g\alpha_i = \Gamma g \Gamma = \bigsqcup_{0 \le i \le p-1} \beta_i g \alpha_i H.$$