Modular Forms: Problem Sheet 9

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16th May 2022

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1. Prove the formula stated in Remark 3.2.29 of the lecture notes: let $f \in \mathcal{M}_k(\Gamma_1(N))$ with q-expansion

$$f(z) = \sum_{m=0}^{\infty} a_m(f) q^m, \ q = e^{2i\pi z},$$

and show that for all $n \in \mathbb{Z}^+$, $T_n(f)$ has Fourier expansion

$$\sum_{m=0}^{\infty} a_m(T_n f) q^m,$$

where

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(\langle d \rangle f).$$

Hint: *Since $\mathcal{M}_k(\Gamma_1(N))$ can be decomposed as a direct sum of eigenspaces of the form $\mathcal{M}_k(\Gamma_1(N), \chi)$, this is tantamount to showing that if $f \in \mathcal{M}_k(\Gamma_1(N), \chi)$, then*

$$a_m(T_n f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f), \ \forall n \in \mathbb{Z}^+. \tag{1}$$

Solution: We assume that $f \in \mathcal{M}_k(\Gamma_1(N), \chi)$ and prove (1) by induction for n taken to be a prime power. Note that the case $n = 1$ is trivial and that we proved the formula for $n = p$ in Theorem 3.2.20. Let us set the convention that $a_{k/l}(f) = 0$ whenever $k/l \notin \mathbb{Z}^+$. This will allow us to consider both the cases $p \mid N$ and $p \nmid N$ at once.

Now consider $n = p^r$, $r \geq 2$ and assume that the formula holds for $n = 1, p, p^2, \ldots, p^{r-1}$. By definition of T_p, we have

$$\begin{aligned} a_m(T_{p^r} f) &= a_m(T_p T_{p^{r-1}} f) - p^{k-1} a_m(\langle p \rangle T_{p^{r-2}} f) \\ &= a_{mp}(T_{p^{r-1}})(f) + \chi(p) p^{k-1} a_{m/p}(T_{p^{r-1}} f) - \chi(p) p^{k-1} a_m(T_{p^{r-2}} f). \end{aligned}$$

Using the induction hypothesis, we obtain

$$\begin{aligned} a_m(T_{p^r f}) = &\sum_{d|(mp, p^{r-1})} \chi(d) d^{k-1} a_{mp^r/d^2}(f) + \chi(p) p^{k-1} \sum_{d|(m/p, p^{r-1})} \chi(d) d^{k-1} a_{mp^{r-2}/d^2}(f) \\ &- \chi(p) p^{k-1} \sum_{p|(m, p^{r-2})} \chi(d) d^{k-1} a_{mp^{r-2}/d^2}(f). \end{aligned} \tag{2}$$

Note that the first term equals

$$a_{mp^r}(f) + \sum_{\substack{d|(mp, p^{r-1}) \\ d>1}} \chi(d) d^{k-1} a_{mp^r/d^2}(f).$$

Rearranging the second term above shows that it equals minus the third term of (2), and therefore that

$$a_m(T_{p^r}f) = a_{mp^r}(f) + \chi(p)p^{k-1} \sum_{d|(m/p,p^{r-1})} \chi(d)d^{k-1}a_{mp^{r-2}/d^2}(f).$$

By distributing $\chi(p)p^{k-1}$ above and reindexing to instead be summing over $\{\tilde{d} = pd \mid (m, p^r)\}$, we obtain the formula.

Now, let $n_1, n_2 \in \mathbb{Z}^+$ such that $\gcd(n_1, n_2) = 1$. Then,

$$\begin{aligned} a_m(T_{n_1}(T_{n_2}(f))) &= \sum_{d|(m,n_1)} \chi(d)d^{k-1}a_{mn_1/d^2}(T_{n_2}f) \\ &= \sum_{d|(m,n_1)} \chi(d)d^{k-1} \sum_{e|(mn_1/d^2,n_2)} \chi(e)e^{k-1}a_{mn_1n_2/d^2e^2}(f). \end{aligned}$$

Once again rearranging the above series and using the fact that $\gcd(n_1, n_2) = 1$ to merge the two sums as one series indexed by $\{l = de \mid (m, n_1n_2)\}$ finishes the proof.

2. Let $f \in \mathcal{M}_k(\Gamma_1(N), \chi)$ be a normalized Hecke eigenform and define its L-function to be the series

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f)n^{-s}.$$

This is a well-defined function when the real part of $s \in \mathbb{C}$ is large enough.

Express $L(f, s)$ as an Euler product, i.e. find an expression of $L(f, s)$ of the form

$$L(f, s) = \prod_{p \text{ prime}} L_p(f, s),$$

where $L_p(f, s)$ is a complex function that depends on p, s, and the Fourier coefficient $a_p(f)$.

Hint: *Use the intrinsic characterization of Hecke eigenforms proved in Proposition 3.2.34.*

Solution: From *ii.* in Proposition 3.2.34, we know that a normalized Hecke eigenform is multiplicative. Hence, we can rewrite $L(f, s)$ as

$$\prod_{p, \text{ prime}} (1 + a_p(f)p^{-s} + a_{p^2}(f)p^{-2s} + \cdots).$$

Moreover, condition *iii.* tells us that the series $\sum_{r=0}^{\infty} a_{p^r}(f)x^r$ equals

$$\frac{1}{1 - a_px + \chi(p)p^{k-1}x^2}.$$

To see this, first note that

$$iii. \implies a_{p^r} - a_pa_{p^{r-1}} + \chi(p)p^{k-1}a_{p^{r-2}} = 0.$$

Hence, after some rearranging,

$$\begin{aligned} \sum_{r=0}^{\infty} a_{p^r}x^r(1 - a_px + \chi(p)p^{k-1}x^2) &= a_1 + a_px - a_pa_1x \\ &= 1, \text{ since we assumed } f \text{ is normalized.} \end{aligned}$$

We conclude that

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f)n^{-s} = \prod_{p, \text{ prime}} \frac{1}{1 - a_pp^{-s} + \chi(p)p^{k-1-2s}}.$$