Modular Forms: Problem Sheet 3

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1. (a) Show that the Eisenstein series E_4 has a simple zero at $z = \rho$ and no other zeros.
   (b) Similarly, show that E_6 has a simple zero at $z = i$ and no other zeros.

> **Solution:** See the solution to Problem Sheet 2, exercise 3.

2. Define the modular invariant $j(z) = \frac{E_4^3}{\Delta}$. *(This is an extremely important invariant, with applications to the elliptic curves and class field theory. The coefficients in its q-expansion are famous for their role in the* ***moonshine conjecture****, which links them to the representation theory of the monster group.)*

   (a) Prove that j is a weakly modular function of weight 0.

> **Solution:** For any $\gamma =\in \mathrm{PSL}_2(\mathbb{Z})$, we compute
> $$j(\gamma \circ z) = \frac{E_4^3(\gamma \circ z)}{\Delta(\gamma \circ z)} = \frac{(cz+d)^{12} E_4^3(z)}{(cz+d)^{12}\Delta(z)} = j(z).$$

   (b) Show that j is holomorphic on $\mathcal{H}$ and has a simple pole at ∞.

> **Solution:** This follows directly from the basic properties of quotients of meromorphic functions applied to $j(z)$ for the statement on $\mathcal{H}$, and applied to $j(e^{2\pi i z})$ for the statement at ∞.

   (c) Show that j induces a bijection $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H} \cong \mathbb{C}$.

> **Solution:** From (a), we see that $j(z)$ doesn't depend specifically on z but rather on the orbit of z under the action of $\mathrm{PSL}_2(\mathbb{Z})$. Hence, j induces a well-defined map from $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$ to $\mathbb{C}$, mapping each orbit to the image of one of its representatives.
> Now, showing that this induced map is bijective is equivalent to showing that for any $\xi \in \mathbb{C}$, the map induced by the auxiliary function $g_\xi := j(z) - \xi$ has a unique zero on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$. We write
> $$g_\xi(z) = \frac{E_4^3(z) - \xi\Delta(z)}{\Delta(z)},$$
> and denote the numerator by h_ξ. The map h_ξ is clearly holomorphic on $\mathcal{H}$ and at ∞. In fact, since Δ vanishes at ∞ and E_4^3 doesn't, we additionally know that $v_\infty(h_\xi) = 0$. Therefore, the valence formula yields
> $$\frac{1}{2}v_i(h_\xi) + \frac{1}{3}v_\rho(h_\xi) + \sum_{P\in W} v_P(h_\xi) = 1,$$
> where W is defined as usual. Since all the vanishing orders on the LHS are positive, you can check that the only solutions are for h_ξ to vanish with order 3 at ρ or for h_ξ to vanish

with order 2 at i or for h_ξ to vanish with order 1 at a unique $P \in W$. In any case, the map induced by h_ξ vanishes exactly once on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathcal{H}$. This implies that the same holds for the map induced by g_ξ and finishes the proof.

3. Show that for any $k \geq 0$, we have

$$|\{(a,b) \in \mathbb{Z}^2_{\geq 0} : 4a + 6b = k\}| = \dim M_k.$$

Solution: Remember that

$$\dim(M_k) = \begin{cases} \lfloor k/12 \rfloor + 1, & k \not\equiv 2 \mod 12 \\ \lfloor k/12 \rfloor, & k \equiv 2 \mod 12 \end{cases}$$

We solve the problem using induction in steps of 12.

- We claim that the cases $k < 0$, k is odd and $k \in \{0, 2, 4, 6, 8\}$ are an easy check.
- Let $k \geq 12$ and note that

$$\dim(M_{k-12}) = \begin{cases} \lfloor k/12 \rfloor, & k \not\equiv 2 \mod 12 \\ \lfloor k/12 \rfloor - 1, & k \equiv 2 \mod 12 \end{cases}$$

In any case, we reduced the problem to showing that the equation

$$4a + 6b = k \tag{1}$$

has always one more solution in $(\mathbb{Z}_{\geq 0})^2$ than the equation

$$4a + 6b = k - 12. \tag{2}$$

Let us consider the map

$$\begin{array}{ccc} \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ (a,b) & \mapsto & (a, b+2) \end{array}$$

This maps solutions of (2) to solutions of (1). The only solutions of (1) that might not be in the image are those with $b < 2$. We show that such a solution always exists.

Let (a', b') be a solution of (1) (we can always pick one since the image of the map is non-empty for all $k \geq 12$, except for $k = 14$, but in this case we can obviously find a solution). We have that $6b'$ is congruent to 0 or 2 modulo 4. Hence there exists a unique positive integer q and some $r \in \{0, 2\}$ such that $6b' = 4q + r$. If $r = 0$, then $(a' + q, 0)$ is a solution of (1). Otherwise, if $r = 2$, then $(a' + q - 1, 1)$ is a solution of (1). Note also that by looking at k modulo 4, we see that there cannot be a solution of (1) with $b = 0$ and another one with $b = 1$. This concludes the proof.

4. Let n be a positive integer.

   (a) Show (by quoting an appropriate theorem from your notes) that the dimension of M_{4n} is $1 + j$, where $j = \lfloor n/3 \rfloor$. Hence show that the functions

$$E_4^n, E_4^{n-3}\Delta, \ldots, E_4^{n-3j}\Delta^j$$

   are a basis of M_{4n}.

Solution: By theorem 1.6.10,

$$\dim(M_k) = \begin{cases} \lfloor k/12 \rfloor + 1, & k \not\equiv 2 \mod 12 \\ \lfloor k/12 \rfloor, & k \equiv 2 \mod 12 \end{cases}$$

Since $k = 4n$ is not congruent to 2 mod 12, we are always in the first case. We proceed by induction in steps of 12. As before, we will let you check the cases $n \in \{0, 1, 2\}$. Assume now that $n \geq 3$. We have to show that the family

$$\mathcal{B} := \{E_4^{n-3i} \Delta^i \mid 0 \leq i \leq j\}$$

is linearly independent. First, note that there cannot be any non-trivial vanishing linear combination of elements of $\mathcal{B}$

$$\sum_{i=0}^{j} \alpha_i E_4^{n-3i} \Delta^i$$

in which $\alpha_0 \neq 0$. Indeed, such a combination doesn't vanish at ∞ since every non-trivial power of Δ does but non of the powers of E_4 do. Moreover there cannot be any non-trivial vanishing linear combination of elements of $\mathcal{B}' := \mathcal{B} \setminus \{E_4^n\}$. Indeed, factorising such a combination by Δ yields a non-trivial vanishing linear combination of elements of

$$\mathcal{B}'' := \{E_4^{n-3i} \Delta^{i-1} \mid 1 \leq i \leq j\}.$$

But by induction hypothesis, this is a basis for M_{4n-12}, so this cannot happen. We conclude that $\mathcal{B}$ is a linearly independent family with cardinality $1 + j$ and therefore forms a basis.

(b) Let $M_{4n}(\mathbb{Z})$ denote the $\mathbb{Z}$-submodule of M_{4n} consisting of modular forms whose q-expansions have integer coefficients. Show that the above functions are a $\mathbb{Z}$-basis of $M_{4n}(\mathbb{Z})$. (You may assume that $\Delta \in M_{12}(\mathbb{Z})$.)

Solution: We recall that

$$E_4 = 1 + 24 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

so $E_4 \in M_4(\mathbb{Z})$ and $E_4^n \in M_{4n}(\mathbb{Z})$ for all $n \geq 0$. Since we assume it is also the case for Δ, all the elements of $\mathcal{B}$ belong to $M_{4n}(\mathbb{Z})$. Assume that

$$f := \sum_{i=0}^{j} \alpha_i E_4^{n-3i} \Delta^i, \ \alpha_i \in \mathbb{C} \ \forall i$$

is an element of $M_{4n}(\mathbb{Z})$. Since the vanishing order of $E_4^{n-3i} \Delta^i$ at ∞ is i, the first coefficient in the q-expansion of f is α_0. Therefore we must have $\alpha_0 \in \mathbb{Z}$. We now let $i \geq 1$ and we assume that $\alpha_k \in \mathbb{Z}$ for $k \leq i - 1$. The coefficient of q^i in the expansion of f is (by assumption) an integer and is given by

$$\alpha_0 b_i^{(0)} + \alpha_1 b_i^{(1)} + \cdots + \alpha_{i-1} b_i^{(i-1)} + \alpha_i,$$

where $b_i^{(k)} \in \mathbb{Z}$ is the coefficient of q^i in the expansion of $E_4^{n-3k} \Delta^k$. This forces $\alpha_i \in \mathbb{Z}$. We conclude that $\alpha_i \in \mathbb{Z}$ for $0 \leq i \leq j$. Since f was an arbitrary element of $M_{4n}(\mathbb{Z})$, this concludes the proof.