Modular Forms: Problem Sheet 5

Sarah Zerbes

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1. Let Γ and Γ' be congruence subgroups such that $\Gamma' \trianglelefteq \Gamma$.

   (a) Show that if $c, d \in C(\Gamma')$ are equivalent in $C(\Gamma)$, then $h_{\Gamma'}(c) = h_{\Gamma'}(d)$.

   Solution: Let $c = [s]_{\Gamma'}$ and $d = [d]_{\Gamma'}$ in $C(\Gamma')$ with $[s]_\Gamma = [t]_\Gamma \in C(\Gamma)$, for $s, t \in \mathbb{P}^1(\mathbb{Q})$. Then there exists $g \in \Gamma$ such that $g \cdot s = t$. Additionally, let $\gamma_s, \gamma_t \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_s \cdot \infty = s$ and $\gamma_t \cdot t = \infty$. We checked in the lecture that

   $$H_{\Gamma',c} = \gamma_s^{-1}\Gamma'\gamma_s \cap \mathrm{SL}_2(\mathbb{Z})_\infty$$

   does not depend on the choice of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \infty = s$. Hence,

   $$\begin{aligned} H_{\Gamma',c} &= (g^{-1}\gamma_t)^{-1}\Gamma'(g^{-1}\gamma_t) \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ &= \gamma_t^{-1}\Gamma'\gamma_t \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ &= H_{\Gamma',d}, \end{aligned}$$

   where we used that Γ' is normal in Γ to pass from the first line to the second. We conclude that c and d have the same width in Γ', by definition of the width of a cusp.

   (b) Let $c \in \mathrm{Cusps}(\Gamma')$. Show that

   $$\sum_{\substack{d \in C(\Gamma') \\ d = c \text{ in } \mathrm{Cusps}(\Gamma)}} h_{\Gamma'}(d) = [\overline{\Gamma} : \overline{\Gamma'}] h_\Gamma(c).$$

   Solution: We use Proposition 2.2.17 with $G = \overline{\Gamma}$, $H = \overline{\Gamma'}$ and $X = \{d \in C(\Gamma') \mid d = c \bmod \Gamma\}$. We obtain

   $$\sum_{d \in X} [\overline{\Gamma}_d : \overline{\Gamma'}_d] = [\overline{\Gamma} : \overline{\Gamma'}]. \tag{1}$$

   Each of the summands in the LHS above equals

   $$\frac{[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma'}_d]}{[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}_d]}.$$

   The numerator is constant for all $d \in X$ since any two elements of X are equivalent mod Γ. More precisely, its value is $h_\Gamma(c)$ for all $d \in X$. We then compute that the LHS of (1) equals

   $$\frac{1}{h_\Gamma(c)} \sum_{d \in X} h_{\Gamma'}(d)$$

   and it concludes the proof.

   (c) Hence show that for p odd, $\Gamma_1(p)$ has exactly $p - 1$ cusps.

Solution: From the lectures we know that any cusp of $\Gamma_1(p)$, which we will denote Γ_1, is either equivalent to $[\infty]$ or to $[0]$ mod $\Gamma_0(p)$, which we will denote Γ_0. In order to use (a) and (b), we first need to show that Γ_1 is normal in Γ_0 and we want to compute its index. To this end note that Γ_1 is the kernel of the surjective map

$$\begin{array}{rccc} \varphi: & \Gamma_0 & \to & (\mathbb{Z}/p\mathbb{Z})^* \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & d \end{array}$$

This shows that Γ_1 is normal in Γ_0 and that $[\Gamma_0 : \Gamma_1] = p-1$. Now, since $-\mathrm{Id} \in \Gamma_0 \setminus \Gamma_1$, we compute $[\overline{\Gamma_0}; \overline{\Gamma_1}] = \frac{p-1}{2}$. We now apply the formula proven in (b) to the set of cusps of Γ_1 that are equivalent to ∞ mod Γ_0. We obtain

$$h_{\Gamma_0}(\infty)\frac{p-1}{2} = \sum_{\substack{d \in C(\Gamma_1) \\ d = [\infty] \mathrm{mod} \Gamma_0}} h_{\Gamma_1}(d) = N_\infty h_{\Gamma_1}(\infty),$$

where N_∞ is the number of such cusps and where we used (a) to obtain the second equality. We showed in the lectures that $h_{\Gamma_0}(\infty) = 1$, and using a similar argument we can show that $h_{\Gamma_1}(\infty) = 1$. We deduce that $N_\infty = \frac{p-1}{2}$.
We proceed similarly for the set of cusps of Γ_1 that are equivalent to $[0]$ mod Γ_0 and conclude.

2. (a) Show that $\mathrm{SL}_2(\mathbb{Z})$ contains an index 2 subgroup Γ which is congruence of level 2.

Solution: We are looking to find $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(2) \subset \Gamma$ and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 2$. We first note that

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) \cong \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$$

is of order 6. Since 3 is a prime divisor of $|\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})|$, it must contain an element of order 3 by Cauchy's theorem (you can also easily find by hand such an element). We denote such an element $\overline{g}$, and let g be a lift of $\overline{g}$ in $\mathrm{SL}_2(\mathbb{Z})$. Defining $\Gamma := \langle \Gamma(2), g \rangle$, we have

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(2)]/[\Gamma : \Gamma(2)] = 2.$$

(b) Show that the only cusp of Γ is $[\infty]$. What is its width?

Solution: Let the element g taken above be the lift $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ of $\overline{g} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Note that since Γ is of index 2 in $\mathrm{SL}_2(\mathbb{Z})$, it is a normal congruence subgroup. We are then in the setting of problem 1 and have

$$N_\infty h_\Gamma([\infty]) = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}],$$

where we let N_∞ be the number of cusps of Γ that are equivalent to $[\infty]$ mod $\mathrm{SL}_2(\mathbb{Z})$, which is simply the number of cusps at level Γ. Since $-\mathrm{Id} \in \Gamma(2)$, we have

$$[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 2.$$

Moreover, as $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(2)$, we have $h_\Gamma([\infty]) < 2$. The width of $[\infty]$ at level Γ is 1 if and only if one of

$$\pm\,\mathrm{Id} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

belongs to Γ, or equivalently, if one of their representatives mod $\Gamma(2)$ belongs to $\langle \overline{g} \rangle$. Since it is not the case, we conclude that $h_\Gamma([\infty]) = 2$ and therefore $N_\infty = |C(\Gamma)| = 1$.

3. Show that the cusp $c = [1/2]$ of $\Gamma_1(4)$ is irregular, and find a generator of the corresponding subgroup H_c.

Solution: We start by finding a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \infty = \frac{1}{2}$. Here, we let $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. In order to find H_c, we now compute that for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4)$, we have

$$\gamma^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma = \begin{pmatrix} a + 2b & b \\ c - 2a + 2d - 4b & d - 2b \end{pmatrix} =: A.$$

If A is to be in H_c, it has to act trivially on ∞. Hence we must have

$$c - 2a + 2d - 4b = 0.$$

This directly implies that $a + 2b = d - 2b \in \{\pm 1\}$.

- Assume first that $1 = d - 2b = a + 2b$ and note that $d - 2b \equiv a + 2b \equiv 1 + 2b \mod 4$. Then $2b \equiv 0 \mod 4$, so there is some $k \in \mathbb{Z}$ such that $b = 2k$ and
$$A = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}.$$
Conversely, we check that for any matrix of this form with $k \in \mathbb{Z}$, there exist a, b, c and d such that
$$\begin{cases} a + 2b = 1 \\ b = 2k \\ c - 2a = 2d - 4b = 0 \\ d - 2b = 1 \end{cases} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4).$$

- Assume now that $-1 = d - 2b = a + 2b$. We still have $d - 2b \equiv a + 2b \equiv 1 + 2b \mod 4$, which implies $2b \equiv 2 \mod 4$. So, there exists some $k \in \mathbb{Z}$ such that $b = 2(2k+1)$ and
$$A = \begin{pmatrix} 1 & 2(2k+1) \\ 0 & 1 \end{pmatrix}.$$
Similarly as above, we check the converse statement, i.e. we check that for any matrix A as above there exists a matrix $B \in \Gamma_1(4)$ so that $\gamma^{-1} B \gamma = A$.

We conclude that

$$H_c = \left\{ (-1)^t \left(\begin{smallmatrix} 1 & 2t \\ 0 & 1 \end{smallmatrix}\right) \mid t \in \mathbb{Z} \right\},$$

which shows that $[1/2]$ is irregular, and we read off it that the width of $[1/2]$ in $\Gamma_1(4)$ is 2.

4. Let Γ and Γ' be congruence subgroups such that $\Gamma' \trianglelefteq \Gamma$. Let f be a meromorphic function on $\mathcal{H}$ that is weakly modular of weight k for Γ. Let $P' \in \mathrm{Cusps}(\Gamma')$, and let P be its image in $\mathrm{Cusps}(\Gamma)$. Then f is holomorphic at P if and only f (viewed as a weakly modular function of weight k for Γ') is holomorphic at P'. Also show that f vanishes at P if and only if f vanishes at P'.

Solution: Let $s, t \in \mathbb{P}^1(\mathbb{Q})$ such that $[t]_{\Gamma'} = P'$, $[s]_{\Gamma'} = P = [s]_{\Gamma}$. Since s and t are congruent mod Γ, there exists some $g \in \Gamma$ such that $g \cdot t = s$. Define $\gamma_s, \gamma_t \in \mathrm{SL}_2(\mathbb{Z})$ as usual. Then

$$v_{P,\Gamma'}(f) = v_{\infty, \gamma_s^{-1}\Gamma'\gamma_s}(f|_k \gamma_s) = v_{\infty, (g\gamma_t)^{-1}\Gamma'(g\gamma_t)}(f|_k g\gamma_t) = v_{\infty, \gamma_t^{-1}\Gamma'\gamma_t}(f|_k \gamma_t),$$

where we used that Γ' is normal in Γ and that f is weakly modular of weight k with respect to Γ to obtain the last equality. We deduce from this that, as a weakly modular function of level Γ', the vanishing orders of f at P and at P' coincide. Moreover, passing from level Γ' to level Γ might modify the width of the cusp P but not the fact that f is holomorphic nor whether or not f vanishes at P. Indeed, the order of vanishing of f at P at level Γ (resp. Γ') is defined as the order of vanishing of $f|_k\gamma_s$ at $[\infty]$ at level $\gamma_s^{-1}\Gamma\gamma_s$ (resp. $\gamma_s^{-1}\Gamma'\gamma_s$). Comparing both Fourier expansions as given in the proof of Lemma 2.5.1., we obtain the result.