Modular Forms: Problem Sheet 6

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1. Show that if f is a non-zero weakly modular function of weight k, level Γ, and $g \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$v_P(f|_k g) = v_{gP}(f)$$

for all $P \in \mathrm{Cusps}(\Gamma)$.

Solution: As explained in the exercise class, this exercise aims to prove that the cusp part of

$$V_\Gamma(f) = \sum_{z \in \Gamma\backslash\mathcal{H}} \frac{v_z(f)}{n_\Gamma(z)} + \sum_{P \in C(\Gamma)} v_P(f)$$

equals that of $V_{g^{-1}\Gamma g}(f|_k)$. So, we want to show that

$$v_{P,g^{-1}\Gamma g}(f|_k) = v_{gP,\Gamma}(f).$$

Let $P = [t]$, $t \in \mathbb{P}^1(\mathbb{Q})$, and let $\gamma_t \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_t \cdot \infty = t$.
First note that

$$\begin{array}{ccc} C(g^{-1}\Gamma g) & \to & C(\Gamma) \\ P = [t] & \mapsto & gP = [gt] \end{array}$$

is a well-defined bijective map.
Then we compute that

$$\begin{aligned} v_{P,g^{-1}\Gamma g}(f|_k g) &= v_{\infty,\gamma_t^{-1}g^{-1}\Gamma g\gamma_t}(f|_k(g\gamma_t)) \\ v_{gP,\Gamma}(f) &= v_{\infty,(g\gamma_t)^{-1}\Gamma g\gamma_t}(f|_k(g\gamma_t)). \end{aligned}$$

2. Show that for the function $F(z) = \prod_{i=1}^{d}(f|_k g_i)(z)$ defined in the proof of Theorem 2.6.3, we have

$$V_{\Gamma'}(F) = \sum_{i=1}^{d} V_{\Gamma'}(f|_k g_i).$$

Solution: let $\in \mathcal{H}$. it is clear (writing the Laurent expansion) that

$$v_z(fg) = v_z(f) + v_z(g).$$

Hence,

$$\sum_{z \in \mathcal{H}} \frac{v_z(F)}{n_\Gamma(z)} = \sum_{z \in \Gamma\backslash\mathcal{H}} \frac{\sum_{i=1}^{d} v_z(f|_k g_i)}{n_\Gamma(z)} = \sum_{i=1}^{d} \sum_{z \in \Gamma\backslash\mathcal{H}} \frac{v_z(f|_k g_i)}{n_\Gamma(z)}.$$

The exchange of the sums cam be justified by a purely complex-analytic argument proving the finiteness of the number of zeroes of f (it is also justified a posteriori since after switching the order, the double sum converges absolutely by the valence formula). We proceed similarly with the cusp part (we can exchange the order of summation since the number of cusps is finite). This proves the statement.

3. Let $F(z) = E_4(2z)$, so $F \in M_4(\Gamma_0(2))$.

(a) Show that $F(\infty) = 1$ and $F(0) = \frac{1}{16}$.

Solution: You can find the Sage code I used to check my computations in the appendix. Let $\Gamma = \Gamma_0(2)$. If we denote $\sum_{n=0}^{\infty} a_n q^n$ the q-expansion of E_4 at ∞, we have that $F(z) = E_4(2z)$ expands as $\sum_{n=0}^{\infty} a_n q^{2n}$. Therefore, $F(\infty) = E_4(\infty) = 1$.
Let us compute the expansion of F at $[0] \in C(\Gamma)$. Letting $\gamma_0 := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ such that $\gamma_0 \cdot \infty = 0$, we want to compute the value of $F|_4\gamma_0$ at ∞ and at level $\gamma_0^{-1}\Gamma\gamma_0$. We have

$$F|_4\gamma_0(z) = z^{-4}F(-1/z) = z^{-4}E_4(-2/z).$$

Letting $z' = z/2$, we obtain

$$F|_4\gamma_0(z) = \frac{1}{16}(z')^{-4}E_4(-1/z') = \frac{1}{16}E_4(z') = \frac{1}{16}\sum_{n=0}^{\infty} a_n q^{1/2.}$$

We deduce $F([0]) = \frac{1}{16}E_4([0]) = \frac{1}{16}$.

(b) Hence show that the subspace of $M_8(\Gamma_0(2))$ spanned by E_4^2, E_4F and F^2 is 3-dimensional, and contains a unique cusp form f with $a_1(f) = 1$. Calculate the q-expansion of this form as far as the q^3 term.

Solution: Let

$$G(z) = \alpha_0 E_4^2(z) + \alpha_1 E_4 F(z) + \alpha_2 F^2(z),$$

and assume that $G \equiv 0$. It must then vanish at all the cusps and we obtain

$$\begin{cases} \alpha_0 + \alpha_1 + \alpha_2 & = & 0 \\ \alpha_0 + \frac{1}{16}\alpha_1 \frac{1}{16^2}\alpha_2 & = & 0 \end{cases} \iff \begin{cases} \alpha_0 & = & \frac{1}{16}\alpha_2 \\ \alpha_1 & = & -\frac{17}{16}\alpha_2 \end{cases}$$

Hence any form $f \in \langle E_4^2, E_4F, F^2 \rangle$ is cuspidal if and only if $f \in \mathbb{C} \cdot (\frac{1}{16}E_4^2 - \frac{17}{16}E_4F + F^2)$. Also recall that E_4 has a simple and unique zero at ρ. This implies

$$G(\rho) = \alpha E_4(2\rho)^2.$$

Since $2\rho \not\equiv \rho \mod \mathrm{SL}_2(\mathbb{Z})$, this will vanish if and only if $\alpha = 0$. We conclude that $\langle E_4^2, E_4F, F^2 \rangle$ is three-dimensional.
By looking at the q-expansion of a generic cusp form

$$f_\alpha = \frac{1}{16}\alpha E_4 - \frac{17}{16}\alpha E_4 F = \alpha F^2,$$

we see that $a_1(f_\alpha) = 1$ if and only if $\alpha = \frac{2B_4}{15}$. We let $f = f_{2B_4/15}$ and compute that $f = q - 8q^2 + 12q^3 + \mathcal{O}(q^4)$ around the cusp ∞.

(c) Use the valence formula and its corollaries to show that

i. the functions $\{E_4^2, E_4F, F^2\}$ are a basis of $M_8(\Gamma_0(2))$,

Solution: By Corollary 2.6.9,

$$\dim(M_8(\Gamma)) \leq 1 + \left\lfloor \frac{8 \cdot 3}{12} \right\rfloor = 3.$$

ii. $f(z) = \frac{\Delta(z)+256\Delta(2z)}{E_4(z)}$.

Solution: Since $E_4 = \sum_{n=0}^{\infty} a_n q^n$ doesn't vanish at ∞, we can invert its q-expansion at ∞, i.e. there exists a power series $\tilde{E}_4 = \sum_{n=0}^{\infty} b_n q^n$ such that $E_4 \tilde{E}_4 = 1$. The coefficients of $\tilde{E}_4$ can be computed inductively and have the general form

$$b_j = -\frac{1}{a_0} \left(a_1 b_{j-1} + a_2 b_{j-2} + \cdots + a_j b_0 \right).$$

After a few computations, we obtain

$$(\Delta(z) + 256\Delta(2z))\tilde{E}_4(z) = q - 8q^2 + 12q^3 + \mathcal{O}(q^4).$$

We now use Corollary 2.6.8 and conclude.

A Sage code

```
R.<q> = PowerSeriesRing(QQ)
E_4 = eisenstein_series_qexp(4,20, normalization='constant')
"""the first parameter is the weight, the second one is the number of terms
and the 'constant' indicates we want it to be normalized so that the constant
term equals 1"""
E_6 = eisenstein_series_qexp(6,20, normalization='constant')
Delta = (E_4^3 - E_6^2)/1728
g = (Delta+256*Delta(q^2))/E_4; g
alpha = (2*bernoulli(4))/15
f = (1/16)*alpha*E_4^2 - (17/16)*alpha*E_4*E_4(q^2) + alpha*E_4(q^2)^2; f
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