Modular Forms: Problem Sheet 7

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1. Let $E_2(z)$ be as defined in Definition 1.7.1.

(a) Show that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$(cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) - \frac{6ic}{\pi(cz+d)}.$$

Solution: We let the reader check that the formula holds for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see Serre's "A course in Arithmetic" for the second formula). We will show that if the formula holds for some given $\gamma_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then it holds for the product. We recall that $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z)$. Hence,

$$\begin{aligned} E_2(\gamma_1\gamma_2 z) j(\gamma_1\gamma_2, z)^{-2} &= E(\gamma_1\gamma_2 z) j(\gamma_1, \gamma_2 z)^{-2} j(\gamma_2, z)^{-2} \\ &= \left(E_2(\gamma_2 z) - \frac{6ic}{\pi(c\gamma_2 z + d)}\right) j(\gamma_2, z)^{-2} \\ &= E_2(z) - \frac{6ic'}{\pi(c'z+d')} - \frac{6ic}{\pi(c\gamma_2 z + d)} j(\gamma_2, z)^{-2}. \end{aligned}$$

We let C, D denote the bottom row of $\gamma_1\gamma_2$. We have reduced the proof to showing that

$$\begin{aligned} \frac{c'}{j(\gamma_2, z)} + \frac{c}{j(\gamma_1, \gamma_2 z)} j(\gamma_2, z)^{-2} &= \frac{C}{j(\gamma_1\gamma_2, z)} \\ \iff \quad j(\gamma_2, z) C - j(\gamma_1\gamma_2, z) c' &= c \end{aligned}$$

We let the reader check that this is indeed the case.

(b) For $N \geq 2$, define

$$E_2^{(N)}(z) = E_2(z) - N E_2(Nz).$$

Show that the function $E_2^{(N)}(z)$ is a modular form of weight 2 and level $\Gamma_0(N)$, and determine its values at the cusps $[0]$ and ∞.

Solution: Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then,

$$E_2^{(N)}(z) j(\gamma z) j(\gamma, z)^{-2} = E_2(\gamma z) j(\gamma, z)^{-2} - N E_2\left(\frac{aNz + bN}{(c/N)Nz + d}\right) j(\gamma, z)^{-2}.$$

Since $\gamma' = \left(\begin{smallmatrix} a & bN \\ c/N & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $j(\gamma', Nz) = j(\gamma, z)$, we can apply the transformation formula to the right-hand side above and obtain

$$E_2(z) - \frac{6ic}{\pi j(\gamma, z)} - N\left(E_2(Nz) + \frac{6i(c/N)}{\pi j(\gamma, z)}\right) = E_2^{(N)}(z).$$

We evaluate $E_2^{(N)}$ at the cusp ∞, i.e. we have to compute the first term of the q-expansion of $E_2(z) - NE_2(Nz)$ when z is in a small enough neighbourhood of infinity. We have computed in Lemma 1.7.2 that

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

in a neighbourhood of ∞. Now, letting $z' = Nz$ and reasoning similarly as for $E_2(z)$, we have that

$$E_2(z') = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^{Nn}$$

in a neighbourhood of ∞. Hence, $E_2^{(N)}(\infty) = E_2(\infty) - NE_2(N \cdot \infty) = 1 - N$.

2. Let Γ be a finite-index **even** subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and c a cusp of Γ. Assume that k is an even integer ≥ 4. Show that we have

$$G_{k,\Gamma,c}(z) = \sum_{(m,n)\in S(c)} \frac{1}{(mz-n)^k}$$

where $S(c)$ is the set of pairs $(m,n) \in \mathbb{Z}^2$ such that $\mathrm{HCF}(m,n) = 1$ and the element $\frac{n}{m} \in \mathbb{P}^1(\mathbb{Q})$ lies in the Γ-orbit c. Describe the sets $S(c)$ for each of the three cusps $\{\infty, 0, \frac{1}{2}\}$ of $\Gamma_1(4)$.

Solution: Let $\gamma_c \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_c \cdot \infty = c$. We compute that

$$\begin{aligned} G_{k,\Gamma,c}(z) = G_{k,\gamma_c^{-1}\Gamma\gamma_c,\infty}|_k\gamma_c^{-1}(z) &= \sum_{\gamma\in(\gamma_c^{-1}\Gamma\gamma_c)^+_\infty\backslash(\gamma_c^{-1}\Gamma\gamma_c)} j(\gamma,\gamma_c^{-1}z)^{-k} j(\gamma_c^{-1},z)^{-k} \\ &= \sum_{\gamma\in(\gamma_c^{-1}\Gamma\gamma_c)^+_\infty\backslash(\gamma_c^{-1}\Gamma\gamma_c)} j(\gamma\gamma_c^{-1},z)^{-k} \\ &= \sum_{\gamma\in(\gamma_c^{-1}\Gamma\gamma_c)^+_\infty\backslash(\gamma_c^{-1}\Gamma)} j(\gamma,z)^{-k}. \end{aligned}$$

We define a map

$$\begin{array}{ccc} \gamma_c^{-1}\Gamma & \longrightarrow & S := \left\{(m,n)\in\mathbb{Z}^2 \mid \gcd(m,n) = 1 \text{ and } \frac{-n}{m} \in \Gamma\cdot c\right\} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & (c,d) \end{array}$$

We have to show that this map is well-defined, i.e. that the bottom row of a matrix in $\gamma_c^{-1}\Gamma$ maps to S, that it is surjective, and that it factors through $(\gamma_c^{-1}\Gamma\gamma_c)^+_\infty$.

- Since $\gamma_c^{-1}\gamma \in \mathrm{SL}_2(\mathbb{Z})$ for any $\gamma \in \Gamma$, we must have $\gcd(c,d) = 1$. Now, letting $c = [p/q]$ for $p, q \in \mathbb{Z}$ such that $\gcd(p,q) = 1$, we take $\gamma_c = \left(\begin{smallmatrix} p & -s \\ q & r \end{smallmatrix}\right)$ for $r, s \in \mathbb{Z}$ such that $pr + qs = 1$. Then, if we denote the entries of $\gamma \in \Gamma$ as a, b, c, d, we have

$$\gamma_c^{-1}\gamma = \begin{pmatrix} ra+sc & rb+sd \\ -qa+pc & -qb+pd \end{pmatrix}.$$

We note that

$$-\frac{-qb+pd}{-qa+pc} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} \frac{p}{q} = \left(-\mathrm{Id}\,\gamma^{-1}\right)\frac{p}{q},$$

which belongs to the Γ-orbit of c since we assumed that Γ is even.

- Let $(m,n) \in S$. We want to show that there exists a matrix with bottom row (m,n) in $\gamma_c^{-1}\Gamma$. By definition of S, there exists $\Gamma \in \Gamma$ such that $-\frac{n}{m} = \gamma\frac{p}{q}$. We claim that $(\pm \mathrm{Id})\gamma_c^{-1}(-\mathrm{Id}\,\gamma^{-1})$ has bottom row (m,n). We compute that

$$\gamma_c^{-1}(-\mathrm{Id}\,\gamma^{-1}) = \begin{pmatrix} -dr+sc & rb-sa \\ qd++pc & -qb-pa \end{pmatrix}.$$

  Therefore, $-n/m = \frac{ap+bq}{cp+dq}$. But $\gcd(m,n) = 1$ and since $(ap+bq, cp+dq)$ is the first column of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} p & -s \\ q & r \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the fraction on the RHS above is also reduced. We conclude that

$$-n = \pm(ap+bq) \quad m = \pm(cp+dq)$$

  and that the signs must coincide in both equations, which shows the claim.

- Finally, assume that $g, g' \in \gamma_c^{-1}\Gamma$ have the same bottom row. Then clearly $g'g^{-1} \in \gamma_c^{-1}\Gamma\gamma_c$ and by Proposition 2.7.1 $g'g^{-1} \in \mathrm{SL}_2(\mathbb{Z})_\infty^+$. Hence $g'g^{-1} \in (\gamma_c^{-1}\Gamma\gamma_c)_\infty^+$. Conversely, if $g = g' \mod (\gamma_c^{-1}\Gamma\gamma_c)_\infty^+$, then g and g' have the same bottom row.

We describe the set $S(\infty)$. Assume that $\frac{p}{q} \in \Gamma_1(4)\cdot\infty$. Then

$$\frac{p}{q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c}$$

for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4)$. We claim that for any pair (a,c) such that $\gcd(a,c) = 1$, $a \equiv 1 \bmod 4$, $c \equiv 0 \bmod 4$, there exists a pair $r, z \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a & -s \\ c & r \end{smallmatrix}\right) \in \Gamma_1(4)$. Indeed, there exists a pair of integers r, s such that $ar + cs = 1$. Moreover,

$$1 = \det\left(\begin{smallmatrix} a & -s \\ c & r \end{smallmatrix}\right) \equiv r \bmod 4,$$

which shows the claim. We conclude that

$$S(\infty) = \{(a,c) \mid \gcd(a,c) = 1,\ a \equiv 1 \bmod 4,\ c \equiv 0 \bmod 4\}.$$

3. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$.. Decompose $\Gamma g \Gamma$ into left Γ-cosets, i.e. find $\{\alpha_i \in \mathrm{GL}_2(\mathbb{Q})^+\}$ such that

$$\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma = \bigcup_i \Gamma\alpha_i.$$

Hence show that if f is a modular form of weight k and level Γ, we have

$$f|_k[\Gamma g \Gamma] = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + p^{k-1}f(pz)$$

Solution: Recall Proposition 3.1.4., that showed that $(g^{-1}\Gamma g \cap \Gamma)\backslash\Gamma$ is in bijection with $\Gamma\backslash\Gamma g\Gamma$. We will also use Corollary 3.1.6. that, given a coset decomposition of $(g^{-1}\Gamma g \cap \Gamma)\backslash\Gamma$

$$\Gamma = \bigcup_j (g^{-1}\Gamma g \cap \Gamma)\alpha_j,$$

yields a coset decomposition of $\Gamma\backslash\Gamma g\Gamma$ as

$$\Gamma g \Gamma = \bigcup_j \Gamma g \alpha_j.$$

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We compute that

$$\frac{1}{p}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}$$
$$\Longrightarrow g^{-1}\Gamma g \cap \Gamma = \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid p|b' \right\} = \Gamma^0(p).$$

We want to find coset representatives α_j of $\Gamma^0(p)\backslash\Gamma$. Since the additional condition on $\Gamma^0(p)$ is that p divides the upper right entry, we let $\alpha_j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$ for $0 \le j < p$ and check whether or not they are coset representatives. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then there is some j such that $\gamma \in \Gamma^0(p)\alpha_j \Leftrightarrow \gamma\alpha_j^{-1} \in \Gamma^0(p) \Leftrightarrow p$ divides $b - ja$. We divide this in two cases

- Assume $p \nmid a$, then there exists $r \in \mathbb{Z}$ such that $ar \equiv 1 \bmod p \implies B - bra \equiv 0 \bmod p$. Let j be the representative of br in $\{0, \ldots, p-1\}$.
- Assume now that $p \mid a$. Then $p \nmid b$ as $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \implies p \nmid b - ja$. So, none of our α_j can be a coset representative of such a γ. However,

  $$\gamma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \in \Gamma^0(p).$$

  Hence we can choose our last representative to be $\tilde{\alpha} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

So, we can now find a decomposition of $\Gamma\backslash\Gamma g\Gamma$ by multiplying each of the α_j and $\tilde{\alpha}$ by g. This yields

$$\Gamma g \Gamma = \bigcup_j \Gamma \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \cup \Gamma \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}.$$

The explicit formula for the weight k action of $\Gamma g\Gamma$ on modular forms of weight k and level Γ follows directly.