Topics in PDEs, lecture 2

Introduction to kinetic theory.

The outline of the lecture is the following:

1. Introduction to kinetic theory
2. Vlasov type equations
3. Vlasov - Poisson equation
4. Some properties of Vlasov equations.

References: pages 1-5 of "Mean field kinetic equations" by F. Golse

1. The kinetic theory of gases is a branch of statistical mechanics, that is a theory describing the evolution of mechanical systems formed by a large number of elements. The main goal of statistical mechanics is to describe the macroscopic behaviour of matter in terms of the behaviour of the constituent elements (sometime called particles, molecules...).

Concrete examples of many body systems relevant for kinetic theory: gases, plasmas, galaxies, crowds, particles in a fluid.

Problem: how the observables (for example the temperature of a gas, the pressure etc) emerge from the underlying Newtonian dynamic of the constituent particles?

The different scales of description of a N- body system (for example a gas) can be translated into different types of mathematical models:

MICROSCOPIC model: one tracks the states of all particles in the system individually (typically the system is a N-body Newtonian system).

For the applications we have in mind (gases, plasma) N is far too big for such model to be practical because $N \approx 10^{20}$. Notice that in this case we have as many ODEs as the number of particles.

- initial datum: how to compute it?
- would this detailed information be relevant?

MACROSCOPIC model: describes phenomena that are accessible to experimental observations, for example it studies the evolution in time of observable quantities such as the density, velocity, pressure, temperature of the system.

In this case we usually deal with PDEs of fluid mechanics (Euler equation, Navier-Stokes...).

For gas modeling purposes, in this situation the gas is modeled as a continuum.

Usually the unknown of the equation is a velocity field $u = u(t,x)$, but it can also be a density $\rho = \rho(t,x)$.

MESOSCOPIC scale: this is the scale of phenomena which are not accessible to macroscopic observation but involve a number of particles that is large enough so that statistical effects are relevant.

This is the realm of kinetic theory of gases.

A kinetic model rely on a statistical description of the system under consideration. We will see that the unknown is a probability density.

Basic notions

- Phase space: this is the space of all possible states occurring in a mathematical model of some physical system

If we study a deterministic system described by an evolution equation, then the phase space is the "smallest" space on which the equation determines a unique, well-behaved solution.

In general the phase space of a classical particle obeying Newton's laws is made of positions and velocities (each particle is labeled by (x, v)).

The main idea in kinetic theory is to study the evolution of a statistical distribution over a certain phase space, where the statistical distribution replaces a huge number of identical objects.
The unknown is called repartition function, or probability density and it is a function $f(t,x,v)$ depending on time, positions and velocities.
The key feature is that f depends also on velocities (main difference w.r.t. fluid models where the unknown is a velocity field $u = u(t,x)$).
The unknown $f(t,x,v)$ is not an osservable.

$f(t,x,v)$ = density of particles that at time t are at position x with velocity v.

In a kinetic model a fluid at rest (zero velocity) is replaced by a huge number of particles moving in all directions with great speed.

MICROSCOPIC DESCRIPTION

N particles obeying Newton's laws. Deterministic, reversible dynamic

N very large

N-particle limit

MESOSCOPIC DESCRIPTION

Kinetic equations:

Boltzmann eq., Vlasov equations.

derivation of fluid equations

hydrodynamic limit

MACROSCOPIC DESCRIPTION

Fluid mechanics: Euler, Navier-Stokes equations.

The unknown is a physical observable

Key problems:

1. Rigorous derivation of kinetic PDEs from the system of N ODEs.

2. Well-posedness of the kinetic equations

3. Qualitative analysis, stability, long-time behaviour

We will study a kinetic model describing plasmas.

• What is a plasma? it is a ionised gas consisting of ions and free electrons.

Plasmas can be artificially generated by heating a neutral gas or by applying a strong electromagnetic field to the point when the gas becomes increasingly electrically conductive.

Examples: stars, interplanetary medium, solar wind, lightning. It is also present in many industrial applications.

Kinetic formalism: The state of the system at time t is described by the distribution function $f = f(t,x,v)$ on the single particle phase space.

We assume $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ with $n = 2,3$.

Definition: $\Omega, \mathcal{V}$ measurable subsets of $\mathbb{R}^n$.

$$\mathcal{N}_{\Omega,\mathcal{V}}(t) = \iint_{\Omega\times\mathcal{V}} f(t,x,v)\,dx\,dv$$
= total number of particles in Ω with velocities in $\mathcal{V}$

This relation can be viewed as a definition of distribution function f.

Note that f is not an observable but, thanks to it we can have access to macroscopic observables:

let $\varphi(x,v)$ be a physical quantity, $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$.

$$\Phi_\Omega(t) := \iint_{\Omega\times\mathcal{V}} \varphi(x,v)\, f(t,x,v)\,dx\,dv$$
= corresponding physical quantity for the portion of the system in $\Omega\times\mathcal{V}$

this is now a macroscopic observable

Let us now introduce the momentum and the kinetic energy for the particle system.

m = mass of one particle (we assume all particles to be identical thus we often normalise m to 1)

$$P_\Omega(t) := \iint_{\Omega \times \mathcal{V}} m v f(t,x,v)\, dx\, dv$$ = momentum at time t of the portion of the system in $\Omega \times \mathcal{V}$

($\varphi(v) = m v$ = momentum of 1 particle with velocity v and mass m.)

$$\mathcal{E}_\Omega(t) := \iint_{\Omega \times \mathcal{V}} \frac{1}{2} m |v|^2 f(t,x,v)\, dx\, dv$$ = kinetic energy of the system in $\Omega \times \mathcal{V}$.

($\varphi(v) = \frac{1}{2} m |v|^2$ = kinetic energy of a particle with mass m and velocity v)

Therefore $P_\Omega(t)$, $\mathcal{E}_\Omega(t)$ are the macroscopic observables defined by the distribution function f, that is a statistical quantity at a microscopic scale.

Let us now see that the evolution of f in time can be studied using very general arguments: assume that each particle is subject to an acceleration field $a = a(t,x,v) \in \mathbb{R}^n$. Particle trajectories are solutions of:

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = a(t, X(t), V(t)) \end{cases} \quad \leftarrow \text{ ODE system}$$

We denote by $t \mapsto \big(X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)\big)$ the solution of :

$$\begin{cases} \dot{X}(t) = V(t) & \rightsquigarrow \text{ Characteristic equations} \\ \dot{V}(t) = a(t, X(t), V(t)) \\ X(t_0,t_0,x_0,v_0) = x_0 \, , \quad V(t_0,t_0,x_0,v_0) = v_0 \, . \end{cases}$$

Assume that the transformation $(x_0,v_0) \mapsto \big(X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)\big)$ is volume preserving in the single particle phase space $\mathbb{R}^n \times \mathbb{R}^n$, and that we can neglect the effects of collisions.

For each open subset $\mathcal{O}_{t_0} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ consider:

$$\mathcal{O}_t := \left\{ \big(X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)\big) \text{ s.t. } (x_0,v_0) \in \mathcal{O}_{t_0} \right\}$$

for all $t \geq t_0$.

Since the transformation is volume preserving the total number of particles in $\mathcal{O}_{t_0}$ is equal to the total number of particles in $\mathcal{O}_t$. Thus :

$$\iint_{\mathcal{O}_{t_0}} f(t_0,x_0,v_0)\, dx_0\, dv_0 = \iint_{\mathcal{O}_t} f(t,x,v)\, dx\, dv$$

If $x = X(t,t_0,x_0,v_0)$, $v = V(t,t_0,x_0,v_0)$ in the RHS

$$\int_{\mathcal{O}_{t_0}} f(t_0,x_0,v_0)\, dx_0\, dv_0 = \iint_{\mathcal{O}_{t_0}} f\big(t, X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)\big)\, dx_0\, dv_0$$

(The Jacobian of this transformation is 1). Since the equality holds for each $\mathcal{O}_{t_0} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ we have

$f(t_0,x_0,v_0) = f\big(t, X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)\big)$, $\forall\, t \geq t_0$ $(x_0,v_0) \in \mathbb{R}^n \times \mathbb{R}^n$.

Assume that f is differentiable in its arguments, and the acceleration field is continuous, then one has:

$$\begin{cases} X(t,t_0,x_0,v_0) = x_0 + v_0 \Delta t + o(\Delta t) \\ V(t,t_0,x_0,v_0) = v_0 + a\,\Delta t + o(\Delta t) \end{cases}$$

$\Delta t = t - t_0$

Therefore:

$$\underline{f(t_0,x_0,v_0)} = f(t, X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0)) =$$

$$f(t_0+\Delta t, x_0 + \underbrace{v_0 \Delta t + o(\Delta t)}_{\Delta x}, v_0 + \underbrace{a(t_0,x_0,v_0)\Delta t + o(\Delta t)}_{\Delta v}) =$$

$$= f(t_0,x_0,v_0) + (\Delta t, \Delta x, \Delta v)\cdot(\partial_t f, \nabla_x f, \nabla_v f)(t_0,x_0,v_0) + o(\Delta t, \Delta x, \Delta v) = f(t_0,x_0,v_0) + \partial_t f(t_0,x_0,v_0)\Delta t +$$

$$\nabla_x f(t_0,x_0,v_0)(v_0 \Delta t + o(\Delta t)) + \nabla_v f(t_0,x_0,v_0)\cdot(a\Delta t + o(\Delta t))$$

$$= f(t_0,x_0,v_0) + \Delta t\big(\partial_t f(t_0,x_0,v_0) + v_0\cdot\nabla_x f(t_0,x_0,v_0) + a(t_0,x_0,v_0)\cdot\nabla_v f(t_0,x_0,v_0)\big) + o(\Delta t).$$

Since this relation must hold for all Δt, x_0, v_0 and all t_0 we conclude that the distribution function f must satisfy the PDE:

$$\underline{\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + a(t,x,v)\cdot\nabla_v f(t,x,v) = 0}$$

This PDE is usually called Liouville equation when the vector field $(v, a(t,x,v))$ is Hamiltonian, and Vlasov equation when it is coupled with another eq. modeling the effect of the particle system on $a(t,x,v)$.

Vlasov equations are kinetic equations where each particle is subject to the acceleration field created by all the other particles in the system.

From Newton equations to Vlasov-type equations

$$\underbrace{m_i}\,\underbrace{\ddot{x}_i}(t) = \sum_j \underbrace{F_{j\to i}(t)}$$

$\longrightarrow$ force exerted by the j-th particle on the i-th particle

m_i mass of the i-th particle

$\rightarrow$ acceleration of the i-th particle

$m_i = 1 \quad \forall i = 1, \dots, N$

If the interaction is translation invariant, the force field usually derives from an interaction potential $W : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$$F_{y\to x} = -\nabla W(x-y).$$

Examples :

charge of x

a) $W(x-y) = c\,\dfrac{q_x q_y}{|x-y|^{n-2}}, \quad c>0$ ELECTROSTATIC POTENTIAL

mass of x

b) $W(x-y) = -c\,\dfrac{m_x m_y}{|x-y|^{n-2}}, \quad c>0$ GRAVITATIONAL POTENTIAL

c) essentially any W arises in some physical problem or the other, and we will see that even smooth W leads to relevant problems.

If now we consider the limit as $N \to +\infty$ the sum over j becomes an integral :

$$\ddot{x}(t) = \int F_{y\to x(t)} \cdot \underbrace{\rho(t,y)\,dy}_{\text{density of particles}} = -\int \nabla W(x(t)-y)\,\rho(t,y)\,dy$$

Thus in this setting

$$a(t,x,v) = a(t,x) = -\int \nabla W(x-y)\,\rho(t,y)\,dy$$

$$= -\left(\nabla W * \rho(t)\right)(x)$$

where $\rho(t,x) = \int f(t,x,v)\,dv$ = density of particles.

Vlasov equation:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + F(t,x)\cdot\nabla_v f = 0 \\ F = -\nabla W * \rho, \quad \rho(t,x) = \int f(t,x,v)\,dv \end{cases}$$

When the force field is given by the electrostatic or gravitational potential we have that:

$$W = \pm \frac{c}{|x|^{n-2}}, \quad c>0, \ n\geq 3 \implies \pm\Delta W = \delta_0$$

(Remark: if $n=2$, $W = \pm c\log|x|$; if $n=1$ $W = \pm c|x|$.)

$\Leftrightarrow$ W is the fundamental solution of the Laplacian

therefore $\Delta(W*\rho) = \pm\rho \implies W*\rho = \pm\Delta^{-1}\rho$

$$\implies F = \mp\nabla\Delta^{-1}\rho.$$

If $W*\rho = U \implies \pm\Delta U = \rho$ Poisson equation

and with this notation $F = -\nabla U$.

Vlasov - Poisson

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = 0 \\ F = -\nabla U \\ \Delta U = \pm\rho, \quad \rho = \int f\,dv \end{cases}$$

QUALITATIVE BEHAVIOUR OF VLASOV TYPE EQUATIONS

Invariants and identities

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \\ F = -\nabla W * \rho \;, \quad \rho = \int f \, dv \end{cases}$$

$U = U_t(x) = (W * \rho_t)(x)$ ⇝ I'm using the pedix t to emphasize the time dependence of the potential and of the density itself.

Let us use the following notation:

$$\mathrm{Kin}_t := \iint \frac{1}{2} |v|^2 f(t,x,v) \, dx \, dv \qquad \text{kinetic energy}$$

$$\mathrm{Pot}_t := \iint U_t(x) f(t,x,v) \, dx \, dv = \int U_t(x) \rho_t(x) \, dx$$

$$= \int \underbrace{W(x) * \rho_t(x)}_{U_t(x)} \rho_t(x) \, dx \qquad \text{potential energy}$$

Total energy of the system:

$$\mathcal{E}_t := \mathrm{Kin}_t + \boxed{\frac{1}{2}} \mathrm{Pot}_t$$

↪ this factor is crucial because we consider unordered pairs of particles. Of course ½ can be included in the definition of potential energy.

We will now formally prove that the total energy of the system $\mathcal{E}_t$ is conserved by the Vlasov equation.

CONSERVATION OF THE TOTAL ENERGY:

$$\frac{d}{dt}\mathcal{E}_t = \iint \frac{|v|^2}{2}\partial_t f(t,x,v)\,dx\,dv + \frac{1}{2}\int \underbrace{W(x)*\partial_t \rho_t(x)\,\rho_t(x)}\,dx$$

Property of the convolution

$$+\frac{1}{2}\int W(x)*\rho_t\,\partial_t\rho_t\,dx \qquad \int W(x)*\partial_t\rho(t,x)\,\rho(t,x)\,dx = \int dx\left(\partial_t\rho(t,x)\,W(x)*\rho(t,x)\right)$$

$$= \frac{1}{2}\iint |v|^2\,\underline{\partial_t f(t,x,v)}\,dx\,dv + \int W(x)*\rho(t,x)\,\underline{\partial_t\rho(t,x)}\,dx$$

To find the equation for ρ we start from the equation for f:

$\underline{\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = 0}$ and we note that this is $\underline{\text{equivalent to}}$:

$$\underline{\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v(Ff) = 0}$$

integrating in dv

$$\partial_t \underbrace{\int f\,dv}_{\rho} + \int \mathrm{div}_x(vf)\,dv + \underbrace{\int \mathrm{div}_v(Ff)\,dv}_{=0 \text{ if } f \text{ decays fast enough in } v} = 0$$

$\Rightarrow \underline{\partial_t\rho = -\mathrm{div}_x\int vf\,dv}$. Recall $F(t,x) = -(\nabla W*\rho_t)(x)$

Then: $$\frac{d}{dt}\mathcal{E}_t = \iint\frac{|v|^2}{2}\left[-\mathrm{div}_x(vf) - \mathrm{div}_v(Ff)\right]dx\,dv$$

$$+\int W(x)*\rho\left(-\mathrm{div}_x\int fv\,dv\right)dx = -\iint\frac{|v|^2}{2}\mathrm{div}_x(vf)\,dx\,dv$$

$$-\iint\frac{|v|^2}{2}\mathrm{div}_v(Ff)\,dx\,dv - \int W*\rho\,\mathrm{div}_x\left(\int fv\,dv\right)dx$$

$$= +\cancel{\iint\nabla_x\left(\frac{|v|^2}{2}\right)vf\,dx\,dv} + \iint v\cdot\underline{F(t,x)}\,f(t,x,v)\,dx\,dv$$

$$+\int\underline{(\nabla W*\rho)(x)}\,vf(t,x,v)\,dx\,dv = 0 \qquad \checkmark$$

So we proved the conservation of the total energy:

$$\mathcal{E}_t = \underbrace{\iint \frac{|v|^2}{2} f(t,x,v)\,dx\,dv}_{Kin_t} + \frac{1}{2}\underbrace{\int (W * \rho_t)(x)\,\rho_t(x)\,dx}_{Pot_t}$$

Thus, If $\mathcal{E}_0 \le C < +\infty \Rightarrow \mathcal{E}_t \le C < +\infty$.

Remark: If we consider the Vlasov - Poisson system

$W = \pm \frac{C}{|x|^{n-2}}$ $(n \geq 3)$, so the potential energy

Pot_t is positive in the Coulombian case and it is negative in the Newtonian case.

If we are in the Coulombian case, so $Pot_t \geq 0$ we have that:

$$\mathcal{E}_0 \le C \Rightarrow \mathcal{E}_t \le C \Rightarrow \begin{cases} 1.\ Kin_t \le C \quad \forall t \\ 2.\ Pot_t \le C \quad \forall t \end{cases}$$

$$Pot_t = \int (W * \rho_t)(x)\,\rho_t(x)\,dx = \int U_t(x)\,\rho_t(x)\,dx$$

$$\overset{\Delta U_t = -\rho_t}{=} -\int U_t(x)\,\Delta U_t(x)\,dx = +\int |\underbrace{\nabla U_t(x)}_{F(t,x) = -\nabla U(t,x)}|^2\,dx$$

$$= \int |F(t,x)|^2\,dx \le C$$

Therefore in the Coulombian case from the conservation of energy we deduce two important estimates:

1. $\iint \frac{|v|^2}{2} f(t,x,v)\,dx\,dv \le C$ $\rightsquigarrow$ integrability of the second moments of f
2. $\int |F(t,x)|^2\,dx \le C$ $\rightsquigarrow$ L^2 integrability of the force field that in this case is the electric field

In the Newtonian case the kinetic and potential energy have opposite sign, so the total energy may be bounded even if the kinetic and potential energy are unbounded. The analysis is more complicated

Formal conservation of the Casimirs of the equation:

$$\frac{d}{dt}\iint \beta(f(t,x,v))\,dx\,dv = 0, \quad \beta : \mathbb{R} \to \mathbb{R}, \ \beta \in C^1.$$

$\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = 0$ multiply by $\beta'(f)$

$$\underbrace{\beta'(f)\,\partial_t f}_{\partial_t \beta(f)} + \underbrace{v\cdot\nabla_x f\,\beta'(f)}_{v\cdot\nabla_x \beta(f)} + \underbrace{F\cdot\nabla_v f\,\beta'(f)}_{F\cdot\nabla_v \beta(f)} = 0$$

$$\partial_t \beta(f) + v\cdot\nabla_x \beta(f) + F\cdot\nabla_v \beta(f) = 0$$

Integrating on $\mathbb{R}^n\times\mathbb{R}^n$:

$$\frac{d}{dt}\iint \beta(f)\,dx\,dv + \underbrace{\iint \mathrm{div}_x\left(v\,\beta(f)\right)dx\,dv}_{=0} + \underbrace{\iint \mathrm{div}_v\left(F\beta(f)\right)dx\,dv}_{=0} = 0$$

We are integrating in all $\mathbb{R}^n\times\mathbb{R}^n$ assuming we have decay in f ✓

Corollary: 1. If $\beta(s) = |s|^p$ we recover the conservation of all L^p norms.

2. If $\beta(s) = -s\log s$ we have conservation of the entropy. These conservation laws express the fact that the Vlasov equation induces a transport by a measure preserving flow.

The conservation of the entropy is in sharp contrast with the case of the Boltzmann equation for which the entropy can only increase in time, unless the system is at equilibrium.

The conservation of the entropy reflects the preservation of information about the system.

* The nonlinear Vlasov-equation is time reversible.

Choose $f(0,x,v) = f_0$, let it evolve until time T, then replace $f(T,x,v)$ with $f(T,x,-v)$ and let it evolve again for time T, reverse velocities again and you are back at f_0.

This implies that the Vlasov equation doesn't have any regularizing effect, at least in the usual sense.

* Equilibria: For Boltzmann equation the only possible equilibria are Gaussian. Vlasov eq. has infinitely many shapes of equilibria.

For example: any $f(x,v) = f^\circ(v)$ define a spatially homogeneous equilibrium.