Tools for PDEs

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Contents

These few notes present some definitions and theorems that will be useful during the seminar "*An Introduction to Mean-Field Limits for Vlasov Equations*". These pages are very informal and give an overview of the concepts that we will use. They can also be used as a brief reminder for those who are already familiar with all these concepts.

It is therefore not necessary to already know all this material, but it should be understood well enough to be used during the semester when needed. Of course this list is not exhaustive and other known results of PDEs may be useful for the understanding of the seminar, but they will be quoted in due course.

1 Functional analysis

1.1 Lebesgue spaces ($\mathbb{R}$-valued)

Definition 1.1. For $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^d$, $d \geq 1$, we define the Lebesgue space or L^p-space as

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^p \, dx < +\infty \right\}.$$

Proposition 1.2. For $1 \leq p \leq \infty$, the L^p-space, $\big(L^p(\Omega), \|\cdot\|_{L^p(\Omega)}\big)$, is a Banach space with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Proposition 1.3. For $1 \leq p \leq \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))' = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. However $(L^\infty(\Omega))' \supsetneq L^1(\Omega)$ and $(L^1(\Omega))' = L^\infty(\Omega)$ if $|\Omega| < +\infty$.

Corollary 1.4. $L^2(\Omega)$ is its own dual. In fact $L^2(\Omega)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx.$$

Proposition 1.5 (*Hölder's inequality*)**.** Let $p, q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

We can generalized Hölder's inequality:

- If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

- If $\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}$ and $f_k \in L^{p_k}(\Omega)$. Then

$$\left\| \prod_{k=1}^n f_k \right\|_{L^r(\Omega)} \leq \prod_{k=1}^n \|f_k\|_{L^{p_k}(\Omega)}.$$

Proposition 1.6 (*Interpolation*)**.** Let $1 \leq p \leq r \leq q \leq +\infty$ and $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. If $f \in L^p(\Omega) \cap L^q(\Omega)$. Then $f \in L^r(\Omega)$ and

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^\theta \; \|f\|_{L^q(\Omega)}^{1-\theta}.$$

Remark 1.7. If $|\Omega| < +\infty$ then the $L^p(\Omega)$ spaces are a decreasing sequence for inclusion, namely if $p \leq q$ then $L^q(\Omega) \subseteq L^p(\Omega)$.

Remark 1.8. Heuristic :

1) The smaller the p the more f has to decay at ∞ to belong to $L^p(\Omega)$.

2) The larger the p the more f has to be locally "bounded" to belong to $L^p(\Omega)$.

Examples 1.9.

1) Constant functions only belong to $L^\infty(\Omega)$. The function $f(x) = \frac{1}{x}$ decays fast enough to belong to $L^p((1, +\infty))$ $\forall \, p > 1$ but not enough for $L^1((1, +\infty))$.

2) The function $f(x) = x^{-\frac{1}{p}}$ goes to $+\infty$ as x goes to 0. The greater the p the more "localized" around 0 the explosion is, namely $x^{-\frac{1}{p}} \in L^q((-1, 1))$ $\forall \, q < p$.

1.2 Convolution

Definition 1.10. Two functions f and g, defined almost everywhere (a.e) and measurable on $\mathbb{R}^d$ are called convolutable if for a.e. $x \in \mathbb{R}^d$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^d$. The convolution product of f and g is then

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\ dy, \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Proposition 1.11.

1) The convolution product is commutative and associative.

2) **Young's convolution inequality.**
For $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and r such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \leq p, q, r \leq +\infty$, we have
$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

3) **Regularity and derivatives.**

   a) If $f \in L^1_{loc}(\mathbb{R}^d)$ and $\phi \in C^0_c(\mathbb{R}^d)$, then f and ϕ are convolutable and the convolution, $f * \phi$, is continuous.

   b) If $f \in L^1_{loc}(\mathbb{R}^d)$ and $\phi \in C^k_c(\mathbb{R}^d)$, then $f * \phi \in C^k(\mathbb{R}^d)$ and $\partial^\alpha(f * \phi) = f * \partial^\alpha \phi$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$.

1.3 Strong and weak convergence

Let $(E, \|\cdot\|_E)$ be a Banach space, e.g. $E = L^p$.

Definition 1.12. A sequence $(f_n)_{n\in\mathbb{N}} \subset E$ converges strongly to $f \in E$ if

$$\|f_n - f\|_E \underset{n\to+\infty}{\longrightarrow} 0.$$

Definition 1.13. A sequence $(f_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $f \in E$ if for all $\phi \in E'$ (the dual space) we have

$$\langle f_n, \phi \rangle_E \underset{n\to+\infty}{\longrightarrow} \langle f, \phi \rangle_E.$$

Where $\langle \cdot, \cdot \rangle_E$ is the duality bracket on E. And if $(f_n)_{n\in\mathbb{N}}$ converges weakly to f, we write $f_n \rightharpoonup f$.

Proposition 1.14. Strong convergence implies weak convergence.

Example 1.15. Weak convergence doesn't imply strong convergence.
Let $E = L^2([0, 2\pi])$ and $f_n = e^{inx}$. The dual of $L^2([0, 2\pi])$ is $L^2([0, 2\pi])$ and the duality bracket is given by

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} g(x)\ dx.$$

For every function $\phi \in E' = L^2([0, 2\pi])$, let's compute

$$\langle f_n, \phi \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \overline{f_n(x)} \phi(x)\ dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} \phi(x)\ dx = \widehat{\phi}(n),$$

where $\widehat{\phi}(n)$ is the Fourier coefficient of ϕ. Then by the Riemann Lebesgue Lemma we know that $\widehat{\phi}(n) \underset{n\to+\infty}{\longrightarrow} 0$. Therefore,

$$\langle f_n, \phi\rangle_{L^2} \underset{n\to+\infty}{\longrightarrow} 0,$$

i.e. f_n converges weakly to the function $f = 0$.

But,

$$\|f_n - 0\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f_n(x)|^2 \, dx = \frac{1}{2\pi}\int_0^{2\pi} |e^{inx}|^2 \, dx = 1 \neq 0.$$

Therefore, f_n doesn't converge strongly to 0.

1.4 Weak convergence of measures

Definition 1.16. We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$. A sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ converges weakly to a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ if for every bounded continuous function $\phi \in C_b^0(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \phi(x)\ \mu_n(dx) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \phi(x)\ \mu(dx).$$

1.5 Weak solutions to PDE's

Motivation : Consider the following PDE with $c \in \mathbb{R}^*$,

$$(T)\begin{cases} \partial_t f(t,x) + c\ \partial_x f(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ f_{in}(x) = f(0,x), & x \in \mathbb{R}. \end{cases}$$

Assume that $f_{in} \in C^1(\mathbb{R})$. We can show that there exists a unique solution $f \in C^1(\mathbb{R}_+ \times \mathbb{R})$ to this system (T).

We can express the fact that $f \in C^1$ is a solution to (T) in the three following equivalent ways :

S1) f satisfies for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\partial_t f(t,x) + c\ \partial_x f(t,x) = 0,$$

and for all $x \in \mathbb{R}$, $f(0,x) = f_{in}(x)$.

S2) The function $(t,x) \mapsto \partial_t f(t,x) + c\ \partial_x f(t,x)$ satisfies for all $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \Big(\partial_t f(t,x) + c\ \partial_x f(t,x)\Big)\phi(t,x)\ dt\ dx = 0,$$

and for all $x \in \mathbb{R}$, $f(0,x) = f_{in}(x)$.

S3) f satisfies for all $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$,

$$\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\Big(\partial_t \phi(t,x) + c\ \partial_x \phi(t,x)\Big)\ dt\ dx = 0,$$

and for all $x \in \mathbb{R}$, $f(0,x) = f_{in}(x)$.

Remark 1.17. Note that S3) can be obtained from S2) by integration by parts using the fact that ϕ is compactly supported in $\mathbb{R}_+ \times \mathbb{R}$.

The essential remark which motived the definition of weak solution is the following : expression S3) does not involve the derivatives of f. In fact, there is no reason a priori for a function f that satisfies S3) to be $C^1(\mathbb{R}_+ \times \mathbb{R})$. Note however that if it is not C^1 then the three expressions above are not equivalent anymore.

Definition 1.18. For a given $f_{in} \in L^1_{loc}(\mathbb{R})$, we say that f is a weak solution to the PDE (T) if it satisfies S3).

Proposition 1.19. All strong solutions (in the sense of S1)) are weak solutions.

Remark 1.20. The definition of weak solutions above leads naturally to the definition of weak derivatives since it implies that a function which is not differentiable may be solution to a PDE. In order to define rigorously weak derivatives we should study Distribution Theory but it's beyond the scope of this course. (See for example *Sandro Salsa, Partial Differential Equations in Action: From Modelling to Theory.*) We will see that for our case, that is first order PDE, it's enough to consider weak solution as measure instead of distribution.

2 Useful Theorems and results

Theorem 2.1 (*Change of Variables Formula*)**.**
Let U and V be two open sets in $\mathbb{R}^d$ and let $\phi : V \longrightarrow U$ a C^1-diffeomorphism. Let $f : U \longrightarrow \mathbb{R}$ an integrable function, $f \in L^1(U)$. Then

$$\int_U f(x)\ dx = \int_V f(\phi(y))|\det D(\phi(y))|\ dy.$$

Theorem 2.2 (*Integration by parts*)**.**
Let Ω be an open set of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. Let n be the outward unit normal vector of $\partial\Omega$. Let $F : \Omega \longrightarrow \mathbb{R}^d$ and $\phi : \Omega \longrightarrow \mathbb{R}$ two continuously differentiable functions. Then

$$\int_\Omega \phi\ \mathrm{div}(F)\ dx = \int_{\partial\Omega} \phi\ F \cdot n\ dS(x) - \int_\Omega \nabla\phi \cdot F\ dx.$$

In particular, if $\Omega = \mathbb{R}^d$ and $\phi \in C^1_c(\mathbb{R}^d)$, then we have

$$\int_\Omega \phi\ \mathrm{div}(F)\ dx = -\int_\Omega \nabla\phi \cdot F\ dx.$$

Theorem 2.3 (*Cauchy-Lipschitz Theorem*)**.**
Let $F :\ \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a continuous vector field with respect to t and uniformly Lipschitz with respect to x. Then the Cauchy Problem :

$$(CP) \begin{cases} \dot{X}(t) = F(t, X), \\ X(0) = x, \end{cases}$$

has a unique maximal solution,

$$\begin{array}{rcccl} X & : &]-\alpha, \beta[& \longrightarrow & \mathbb{R}^d \\ & & t & \longmapsto & X(t), \end{array}$$

with $\alpha, \beta > 0$ and such that :

i) (existence) $X(t)$ is solution of (CP).

ii) (uniqueness) if $Y :]-a, b[\longrightarrow \mathbb{R}^d$ is another solution, then $]-a, b[\subset]-\alpha, \beta[$ and for all $t \in]-a, b[$ we have $Y(t) = X(t)$.

iii) (maximality) if $\beta < \infty$ (respectively $\alpha < \infty$), then $\lim_{t\to\beta} |X(t)| = \infty$ (respectively $\lim_{t\to-\alpha} |X(t)| = \infty$).

If $\alpha = \beta = \infty$, $X(t)$ is called a global solution.

Theorem 2.4 (*Grönwall Lemma*).
Let $v : [0, T] \longrightarrow \mathbb{R}^+$ continuous. We suppose there exists two constants $a, b > 0$ such that:

$$v(t) \leq b + a \int_0^t v(s)\, ds \qquad \text{for every } t \in [0, T].$$

Then we have $v(t) \leq e^{at}\, b$, $\quad$ pour tout $t \in [0, T]$.

Poisson equation

Consider the Laplace equation, $\Delta u = \delta_0$, on $\Omega = \mathbb{R}^d$, for $d \geq 3$. The fundamental solution of this PDE is given by

$$G_d(x) = C_d \frac{1}{|x|^{d-2}}, \qquad \forall x \neq 0.$$

Now consider the Poisson equation on $\Omega \subseteq \mathbb{R}^d$,

$$-\Delta u = f.$$

The unique solution to the Poisson equation is given by

$$u(x) = G_d * f(x) = \int_{\mathbb{R}^d} C_d \frac{1}{|x-y|^{d-2}}\, f(y)\, dy.$$