Chapter 2: Transport Equations

2.1 Transport Equations with Constant Coefficients

Let $v \in \mathbb{R}^n \setminus \{0\}$. The transport equation is given by

$$\partial_t f + v \cdot \nabla_x f = 0 \qquad (T)$$

with $f \equiv f(t,x) \in \mathbb{R}$ unknown, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$.

We introduce an important tool called the method of characteristics, which is important for PDE theory.

Definition (Characteristic Curve):

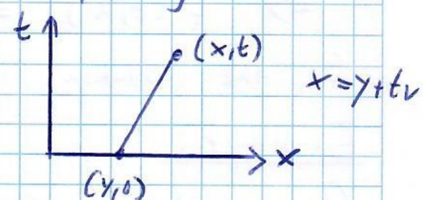
For a given $y \in \mathbb{R}^n$ assume that $\gamma(t)$ solves the ODE

$$(CE) \quad \begin{cases} \dot{\gamma}(t) = v \\ \gamma(0) = y \end{cases}$$

$\gamma(t)$ is called a characteristic curve. Here, $\gamma(t) = y + tv$.

Hence, $\{(t, y+tv) \mid t \in \mathbb{R}\}$ is a straight line passing through $(0, y)$.

if $v > 0$:



If we assume to have a solution $\gamma(t)$ of this ODE associated to the transport operator $\partial_t + v \cdot \nabla_x$, what can we say about a solution of (T)?

Let $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ be a solution of (T). Then,

$$\frac{d}{dt} f(t, \gamma(t)) = \partial_t f(t, \gamma(t)) + \sum_{k=1}^n \partial_{x} f(t, \gamma(t)) \dot{\gamma}_k(t)$$

$$\overset{(CE)}{=} \partial_t f(t, \gamma(t)) + \sum_{k=1}^n v_k \partial_x f(t, \gamma(t))$$

$$= (\partial_t f + v \cdot \nabla_x f)(t, \gamma(t)) \underset{(T)}{=} 0$$

$\Rightarrow f(t, \gamma(t)) = \text{Const.}$

C^1 solutions of (T) are constant along the characteristic curves.

Theorem (Existence & Uniqueness):

Let $f_0 \in C^1(\mathbb{R}^n)$. Then the problem

$$(*) \quad \begin{cases} \partial_t f(t,x) + v \cdot \nabla_x f(t,x) = 0 & x \in \mathbb{R}^n,\ t > 0 \\ f(0,x) = f_0(x) \end{cases}$$

has a unique solution $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$, given by

$$f(t,x) = f_0(x - tv) \qquad x \in \mathbb{R}^n,\ t > 0$$

Proof: **Uniqueness:**

Let $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ be a solution of $(*)$. Then f is constant along $\gamma(t) = y + tv$, i.e.

$$f(t, \gamma(t)) = f(t, y + tv) = f(0,y) = f_0(y)$$

Now substitute $x = y + tv$

$$\Rightarrow f(t,x) = f_0(x - tv), \qquad x \in \mathbb{R}^n,\ t \geq 0.$$

So, the solution is uniquely determined by f_0.

Existence:

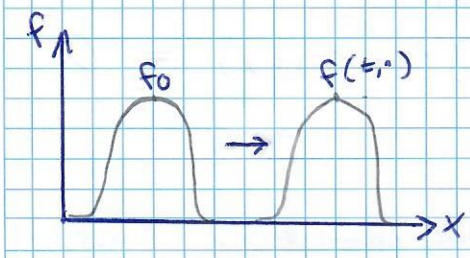
Claim: the function $f: \mathbb{R} \times \mathbb{R}^n \ni (t,x) \longmapsto f_0(x - tv) \in \mathbb{R}$ is C^1 and solves $(*)$.

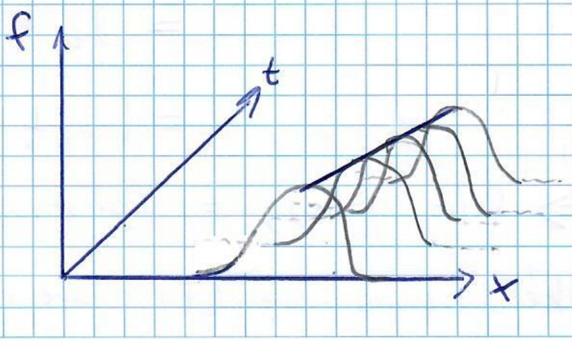
- f is C^1 as a composition of two C^1 functions f_0 & $(t,x) \mapsto x - tv$
- Obviously $f(0,x) = f_0(x) \quad \forall x \in \mathbb{R}^n$
- Compute $\partial_t f(t,x) = \nabla f_0(x - tv)(-v)$

  $\nabla_x f(t,x) = \nabla f_0(x - tv)$

$$\Rightarrow \partial_t f(t,x) + v \cdot \nabla_x f(t,x) = 0 \qquad \square$$

Sketch: $f(t,x) = f_0(x - tv)$





2.2 Transport Equations with Variable Coefficients

Now $V \equiv V(t,x) \in \mathbb{R}^n$ is a time dependent vector field on $[0,T] \times \mathbb{R}^n$.
Hence, our problem becomes:

$$\begin{cases} \partial_t f(t,x) + V(t,x) \cdot \nabla_x f(t,x) = 0, & x \in \mathbb{R}^n,\ 0 < t < T \\ f(0,x) = f_0(x) \end{cases}$$

with $f_0 \in \mathbb{R}$ given, $f \in \mathbb{R}$ unknown.

We need some assumptions on V:

- Each component V_i has partial derivatives wrt. the x_j for $j = 1,..,n$.
- $V \in C([0,T] \times \mathbb{R}^n; \mathbb{R}^n)$ and $\nabla_x V \in C([0,T] \times \mathbb{R}^n; M_n(\mathbb{R}))$ (H1)
- $\exists K > 0$ s.t. $|V(t,x)| \leq K(1 + |x|)$ $\forall (t,x) \in [0,T] \times \mathbb{R}^n$ (H2)

Definition (Characteristic Curve):

Let γ solve

$$(CE) \begin{cases} \dot{\gamma}(s) = V(s, \gamma(s)) \\ \gamma(t) = x \end{cases}$$

The characteristic curve $\{(s, \gamma(s)) \mid s \in [0,T]\}$ is passing through x at time $s = t$.

Now, we have two steps for the method of characteristics:

1) Defining the flow associated to (CE)
2) Using this flow to solve the transport equation.

2.2.1 The Characteristic Flow

We can prove existence, uniqueness & regularity of the solution of the differential system of characteristics:

Theorem:

Assume V satisfies (H1) & (H2). Then $\forall t \in [0,T]$ and $\forall x \in \mathbb{R}^n$ the problem (CE) has a unique C^1 solution on $[0,T]$.
Denote $\gamma(s) =: X(s,t,x)$.

X satisfies:

(i) $X \in C^1([0,T] \times [0,T] \times \mathbb{R}^n ; \mathbb{R}^n)$

(ii) $\partial_s \partial_{x_j} X(s,t,x)$ & $\partial_{x_j} \partial_s X(s,t,x)$ exist $\forall (s,t,x)$, $\forall j = 1,..,N$ and $\partial_s \partial_{x_j} X(s,t,x) = \partial_{x_j} \partial_s X(s,t,x)$, $\partial_s \partial_{x_j} X$ is continuous

(iii) If $V \in C^k$ and $\nabla_x V \in C^k$ then $X \in C^{k+1}$

<u>Proof:</u> (H1) $\Rightarrow$ V satisfies the assumptions for the Cauchy-Lipschitz thm.

$\Rightarrow$ The diff. system of characteristics has a unique C^1 maximal solution γ on some open interval $I(t,x) \subset [0,T]$, s.t. $t \in I(t,x)$.

(maximal solution: the interval $I(t,x)$ cannot be extended to any bigger open interval).

$$\forall s \in I(t,x): \quad |\gamma(s)| \leq |x| + \left| \int_t^s |V(\tau, \gamma(\tau))| \, d\tau \right|$$

$$\overset{(H2)}{\leq} |x| + \left| \int_t^s (1 + |\gamma(\tau)|) \, d\tau \right|$$

$$\text{Gronwall} \longrightarrow \leq (|x| + \mathcal{X} T) e^{\mathcal{X} T}, \quad s \in I(t,x).$$

$$\Rightarrow \sup_{s \in I(s,x)} |\gamma(s)| < \infty \quad \Rightarrow \quad \overline{I(t,x)} = [0,T]$$

Properties (i)–(iii) follow from differentiability properties of the solution of a diff. eq. $\square$

Why is (H2) essential? It may happen that the characteristic curves are not defined:

<u>Example:</u> $n = 1$, $V(t,x) = x^2$. Then the problem

$$\begin{cases} \dot{\gamma}(s) = \gamma(s)^2 \\ \gamma(t) = x \end{cases}$$

has a unique maximal solution

$$\gamma(s) = \frac{x}{1 - (s-t)x}$$

that is only defined for

$$\begin{cases} s < t + \frac{1}{x} & \text{if } x > 0 \\ s > t + \frac{1}{x} & \text{if } x < 0 \end{cases},$$

hence not necessarily on the whole $[0,T]$.

To show the main result on uniqueness & existence of the solution of the transport equation we first need to prove some properties of the flow:

Theorem:

Assume V satisfies (H1) & (H2). Then (denoting $\gamma(s) =: X(s,t,x)$):

(i) X satisfies the flow property: $X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x)$ $\forall x \in \mathbb{R}^n$, $t_1, t_2, t_3 \in [0,T]$

(ii) $\forall s,t \in [0,T]$, $X(s,t,\cdot) : x \longmapsto X(s,t,x)$ is a C^1-diffeomorphism onto itself.

(iii) Let $J(s,t,x) := \det(D_x X(s,t,x))$. Then J is the solution of
$$\begin{cases} \partial_s J(s,t,x) = \operatorname{div}_x V(s, X(s,t,x))\, J(s,t,x), & x \in \mathbb{R}^n, s,t \in [0,T] \\ J(t,t,x) = 1 \end{cases}$$

(iv) $\forall s,t \in [0,T]$ the diffeomorphism $X(s,t,\cdot)$ is orientation preserving.

If $\operatorname{div}_x V(t,x) = 0$ $\forall x \in \mathbb{R}^n$, $t \in [0,T]$, then $\forall \phi \in C_c(\mathbb{R}^n)$
$$\int_{\mathbb{R}^n} \phi(X(s,t,x))\,dx = \int_{\mathbb{R}^n} \phi(x)\,dx \qquad \forall s,t \in [0,T].$$

Proof:

(i) Note that $X(t,t,x) = \gamma(t) = x$. Hence, at $t_3 = t_2$, the curves
$$t_3 \longmapsto X(t_3, t_2, X(t_2, t_1, x))$$
and
$$t_3 \longmapsto X(t_3, t_1, x)$$
are both passing through $X(t_2, t_1, x)$. By uniqueness, both curves must coincide.

(ii) From (i) we get
$$X(t,s,\cdot) \circ X(s,t,\cdot) = X(t,t,\cdot) = id_{\mathbb{R}^n}$$
$$X(s,t,\cdot) \circ X(t,s,\cdot) = X(s,s,\cdot) = id_{\mathbb{R}^n}.$$

Since $X(s,t,\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ as seen before, $X(s,t,\cdot)$ is a C^1-diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, with inverse $X(s,t,\cdot)^{-1} = X(t,s,\cdot)$.

(iii) By the theorem before, we get $\partial_s D_x X = D_x \partial_s X$.

We compute:
$$\begin{aligned}
\partial_s J(s,t,x) &= \partial_s \det(D_x X(s,t,x)) \\
&= D_x \det(D_x X(s,t,x)) \cdot \partial_s D_x X(s,t,x) \\
&\overset{(D\det)(A)\cdot B = \det(A)\cdot \operatorname{trace}(A^{-1}B)}{=} \det(D_x X(s,t,x)) \operatorname{trace}(D_x X(s,t,x)^{-1} D_x \partial_s X(s,t,x)) \\
&\overset{(CE)}{=} J(s,t,x) \operatorname{trace}(D_x X(s,t,x)^{-1} D_x (V(s, X(s,t,x)))) \\
&\overset{\text{chain rule}}{=} J(s,t,x) \operatorname{trace}(D_x X(s,t,x)^{-1} (D_x V)(s, X(s,t,x)) D_x X(s,t,x)) \\
&\overset{tr(AB) = tr(BA)}{=} J(s,t,x) \operatorname{trace}((D_x V)(s, X(s,t,x))) \\
&= J(s,t,x) \operatorname{div}_x V(s, X(s,t,x))
\end{aligned}$$

- $X(t,t,x) = x \Rightarrow D_x X(t,t,x) = I \Rightarrow J(t,t,x) = 1$

(iv) By (iii) we get

$$J(s,t,x) = \exp\left(\int_t^s \operatorname{div}_x V(\tau, X(\tau,t,x)) \, d\tau \right) > 0,$$

hence $X(s,t,\cdot)$ preserves the orientation.

If $\operatorname{div}_x V \equiv 0 \Rightarrow J(s,t,\cdot) = 1 \quad \forall s,t \in [0,T], x \in \mathbb{R}^n$

In this case $\forall \varphi \in C_c(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \varphi \, dx \overset{X \text{ surjective}}{=} \int_{X(s,t,\mathbb{R}^n)} \varphi \, dx \overset{''}{=} \int_{\mathbb{R}^n} \varphi \circ X(s,t,x) \, |\det(D_x X(s,t,x))| \, dx$$

$$= \int_{\mathbb{R}^n} \varphi \circ X(s,t,x) \, dx$$

Therefore, $X(s,t,\cdot)$ is measure preserving

$\square$

2.2.2 Solving the Transport Equation

We are now ready to state the main result on existence and uniqueness of the solution of the transport equation.

Theorem:

Assume V satisfies (H1) and (H2) and $f_0 \in C^1(\mathbb{R}^n)$. Then, the problem

$$\begin{cases} \partial_t f(t,x) + V(t,x) \cdot \nabla_x f(t,x) = 0 & x \in \mathbb{R}^n, 0<t<T \\ f(0,x) = f_0(x) \end{cases}$$

has a unique solution $f \in C^1([0,T] \times \mathbb{R}^n)$, given by

$$f(t,x) = f_0(X(0,t,x)) \qquad \forall t \in [0,T], x \in \mathbb{R}^n$$

Proof: Uniqueness:

We consider the map $t \mapsto f(t, X(t,0,y))$, which is C^1.

Hence
$$\frac{d}{dt} f(t, X(t,0,y)) = \partial_t f(t, X(t,0,y)) + \nabla_x f(t, X(t,0,y)) \partial_s X(t,0,y)$$
$$\overset{(CE)}{=} \partial_t f(t, X(t,0,y)) + \nabla_x f(t, X(t,0,y)) \cdot V(t, X(t,0,y))$$
$$= (\partial_t f + V \cdot \nabla_x f)(t, X(t,0,y)) = 0$$

$$\Rightarrow \text{const} = f(t, X(t,0,y)) = f(0,y) = f_0(y)$$

Now set $x := X(t,0,y) \overset{x\ C^1\text{-diffeo}}{\Longrightarrow} y = X(0,t,x)$

$$\Rightarrow f(t,x) = f_0(X(0,t,x)) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

Existence: Set $f(t,x) = f_0(X(0,t,x))$

Obviously $f \in C^1$ and $f(0,x) = f_0(X(0,0,x)) = f_0(x)$.

It remains to show that f satisfies the transport equation.

Claim: $\partial_t X(s,t,x) + (V(t,x) \nabla_x) X(s,t,x) = 0 \quad \forall s,t \in (0,T)$.

$\hookrightarrow$ Indeed we know that $X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x)$.

The claim follows from taking the derivatives wrt. t_2, using the chain rule & (CE) and setting $t_2 = t_1 = t$, $t_3 = s$.

Compute:

- $\nabla f_0(X(0,t,x)) \cdot \partial_t X(0,t,x) = \partial_t f_0(X(0,t,x))$
- $\nabla f_0(X(0,t,x)) \cdot (V(t,x)\cdot \nabla_x) X(0,t,x) = V(t,x) \cdot \nabla_x (f_0(X(0,t,x)))$

Adding them up leads us to:

$$\partial_t f_0(X(0,t,x)) + V(t,x)\cdot \nabla_x (f_0(X(0,t,x)))$$

$$= \nabla f_0(X(0,t,x)) \left(\partial_t X(0,t,x) + (V(t,x)\cdot \nabla_x) X(0,t,x) \right)$$

$$\overset{\text{Claim}}{=} 0$$

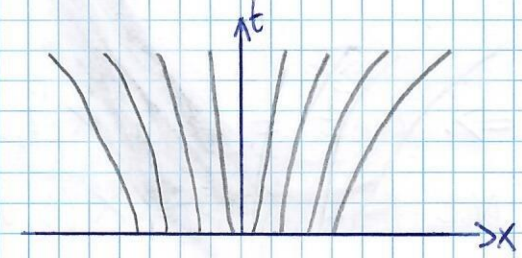
Therefore, $f_0(X(0,t,x))$ satisfies the transport equation. $\square$

Examples for the Method of Characteristics

$$(T) \begin{cases} \partial_t f(t,x) + v(t,x) \nabla_x f(t,x) = 0 \\ f(0,x) = f_0(x) \end{cases}$$

① Let's consider $v(t,x) = x$.

Then
$$(CE) \begin{cases} \dot{\gamma}(t) = \gamma(t) \\ \gamma(0) = x_0 \end{cases}$$



This has solution $\gamma(t) = x_0 e^t$

Substitute $x = x_0 e^t \Rightarrow x_0 = x e^{-t}$

$\Rightarrow$ We have solution $\underline{f(t,x) = f_0(x e^{-t})}$

② If we consider
$$v(t,x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

The characteristics will look like this: