STUDENT SEMINAR: AN INTRODUCTION TO MEAN-FIELD LIMITS FOR VLASOV EQUATIONS

Abstract. In today's[1] talk we focus on the transport equation in conservative form, namely

$$\partial_t \mu + \mathrm{div}_x(\mu V) = 0.$$

We discuss the existence and uniqueness of weak solutions as well as C^1-solutions under more restrictive conditions. The talk follows section 2.3 of [Gol13].

Talk 4: Conservative Transport and Weak Solutions

Adrian Dawid

A. Basic Definitions

Definition 4.1. Let $(X, \Omega_X), (Y, \Omega_Y)$ be measurable spaces, $f : X \to Y$ a measurable function and $\mu : \Omega_X \to \mathbb{R}_{\geq 0}$ a measure on (X, Ω_X). Then the measure $f\sharp\mu : \Omega_Y \to \mathbb{R}_{\geq 0}$ defined by

$$f\sharp\mu(U) \coloneqq \mu(f^{-1}(U)) \forall U \in \Omega_Y$$

is called the *push-forward measure* of μ by f.

Example 4.2. Consider $X = \{0, 1\}$ with the counting measure μ and let $Y = \mathbb{R}$ with the usual Lebesgue measure and consider the measurable function $i : X \hookrightarrow \mathbb{R}$. Then

$$i\sharp\mu(X) = |X \cap \{0, 1\}| \implies i\sharp\mu = \delta_0 + \delta_1.$$

Example 4.3. Consider the Lebesgue-measurable function $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x + \alpha$ for $\alpha \in \mathbb{R}$ and the Lebesgue measure λ on $\mathbb{R}$. Then

$$f\sharp\lambda = \lambda$$

since the Lebesgue measure is translation invariant.

Example 4.4. A more interesting example is given by the following: Let $X = [0, 1] \times [0, 2\pi]$ with μ being the uniform probability measure on X. Let $Y = D^2 \subset \mathbb{R}^2$ be the closed unit disk, then consider the measurable function

$$\begin{aligned} f : X &\to D^2 \\ t, \theta &\mapsto r \cdot (\cos\theta, \sin\theta). \end{aligned}$$

We know that

$$f\sharp\mu(r_2 D^2 \setminus r_1 D^2) = r_2 - r_1$$

for any $0 \leq r_1 \leq r_2 \leq 1$. Indeed we have for any measurable $U \subset D^2$, that

$$f\sharp\mu(U) = \iint_U \frac{1}{2\pi r} dr d\theta,$$

i.e. $f\sharp\mu$ admits a density w.r.t. the Lebesgue measure on D^2.

Proposition 4.5. *Let $(X, \Omega_X), (Y, \Omega_Y)$ be measurable spaces, $T : X \to Y$ a measurable function and $\mu : \Omega_X \to \mathbb{R}_{\geq 0}$ a measure on (X, Ω_X). Let $\nu \coloneqq T\sharp\mu$, then*

$$\varphi \in L^1(Y, \nu) \implies \varphi \circ T \in L^1(X, \mu)$$

with

$$\int_Y \varphi d\nu = \int_X \varphi \circ T d\mu.$$

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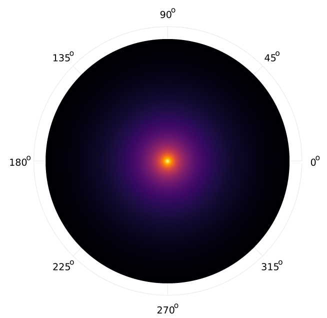


FIGURE 1. The density of $f\sharp\mu$ from example 4.4 on an annulus going from a small radius to radius 1.

Proof. For any measurable set $U \in \Omega_Y$ we have that

$$\mathbb{1}_U \circ T = \mathbb{1}_{T^{-1}(U)},$$

which by construction of ν implies

$$\int_Y \mathbb{1}_U d\nu = \nu(U) = \mu(T^{-1}(U)) = \int_X \mathbb{1}_{T^{-1}(U)} d\mu = \int_X (\mathbb{1}_U \circ T) d\mu.$$

Thus the proposition is true for indicator functions, then by the standard machinery of *measure-theoretic induction*[1] this gives us the proposition for $L^1(Y, \nu)$. □

We can further see a formula for the concrete example of a diffeomorphism and the Lebesgue measure:

Corollary 4.6. *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ almost-everywhere on $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a (C^1-)diffeomorphism. Then*

$$T\sharp(f\lambda) = f \circ T^{-1} \cdot \left|\det\left(DT \circ T^{-1}\right)\right|^{-1} \cdot \lambda,$$

where λ again is the Lebesgue measure on $\mathbb{R}^n$.

Proof. Let $U \subset \mathbb{R}^n$ be a Lebesgue-measurable subset, then

$$T\sharp(f\lambda)(U) = f \cdot \lambda(T^{-1}(U)) = \int_{\mathbb{R}^n} f \cdot \mathbb{1}_{T^{-1}(U)} d\lambda.$$

Since T is a diffeomorphism we can apply the change of variables[2] formula to this integral to obtain the following:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) \cdot \mathbb{1}_{T^{-1}(U)}(x) d\lambda(x) &= \int_{\mathbb{R}^n} f(T^{-1}(x)) \cdot \mathbb{1}_{T^{-1}(U)}(T^{-1}(x)) \cdot \left|\det D_x T^{-1}\right| d\lambda(x) \\
&= \int_{\mathbb{R}^n} f(T^{-1}(x)) \cdot \mathbb{1}_U(x) \cdot \underbrace{\left|\det D_x T^{-1}\right|}_{=\left|\det D_{T^{-1}(x)} T\right|^{-1}} d\lambda(x) \\
&= \int_{\mathbb{R}^n} \left(f(T^{-1}(x)) \left|\det D_{T^{-1}(x)} T\right|^{-1}\right) \cdot \mathbb{1}_U(x) d\lambda(x) \\
&= \left(f \circ T^{-1} \cdot \left|\det\left(DT \circ T^{-1}\right)\right|^{-1} \cdot \lambda\right)(U),
\end{aligned}$$

thus completing the proof. □

Definition 4.7. Let $w\mathcal{M}(\mathbb{R}^n)$ denote the set of Radon measures on $\mathbb{R}^n$ topologized with the weak topology.

[1] i.e. passing to linear combinations of finitely many indicator functions and then by density onto all integrable functions.
[2] here we exchange x with $T^{-1}(x)$

B. Weak Solutions

We will now give the definition of a weak solution to the conservative transport equation.

Definition 4.8. Let $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and let $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$. A *weak solution* to the conservative transport equation

$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mu V) = 0 \\ \mu|_{t=0} = \mu_0, \end{cases}$$

is $\mu \in C([0,T], w\mathcal{M}(\mathbb{R}^n))$ s.t. $\mu|_{t=0} = \mu_0$ and for any $\varphi \in C_c^1((0,T) \times \mathbb{R}^n)$ we have

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t \varphi(t,x) + V(t,x) \cdot \nabla_x \varphi(t,x)) d\mu(t,\cdot)(x) dt = 0.$$

A natural question to ask is, why this is a sensible definition. The answer is given by the following proposition:

Proposition 4.9. *Let $f \in C^1([0,T] \times \mathbb{R}^n)$, then*

$$\partial_t f + \operatorname{div}_x(fV) \equiv 0 \iff \int_0^T \int_{\mathbb{R}^n} (\partial_t \varphi(t,x) + V(t,x) \cdot \nabla_x \varphi(t,x)) f(t,x) dx dt = 0 \forall \varphi \in C_c^1((0,T) \times \mathbb{R}^n).$$

We can easily see that this equation is equivalent to the weak formulation for $\mu(t,\cdot) = f(t,\cdot) \cdot \lambda$ with λ being the Lebesgue measure. So now let us prove the proposition.

Proof. We start by defining the following vector field:

$$W : [0,T] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$$
$$(t,x) \mapsto \begin{cases} (\varphi(t,x) \cdot f(t,x), \varphi(t,x) \cdot f(t,x) \cdot V(t,x)) & \text{if } t \in (0,T) \\ 0 & \text{if } t \in \{0,1\}. \end{cases}$$

By construction this vector field is in $C_c^1((0,T) \times \mathbb{R}^n, \mathbb{R} \times \mathbb{R}^n)$. Thus it vanishes on the boundary which by Green's formula gives us

$$\begin{aligned} 0 &= \int_{(0,T) \times \mathbb{R}^n} \operatorname{div}_{t,x}(W(t,x)) dx dt \\ &= \int_{(0,T) \times \mathbb{R}^n} \operatorname{div}_{t,x}(f(t,x) \cdot \varphi(t,x), f(t,x) \cdot \varphi(t,x) \cdot V(t,x)) dx dt. \end{aligned}$$

Now we take a more exact look at the expression inside the integral and obtain:

$$\begin{aligned} &\operatorname{div}_t x,(f(t,x) \cdot \varphi(t,x), f(t,x) \cdot \varphi(t,x) \cdot V(t,x)) \\ &= \partial_t(f(t,x)\varphi(t,x)) + \operatorname{div}_x(f(t,x)\varphi(t,x)V(t,x)) \\ &= \partial_t(f(t,x)\varphi(t,x)) + \varphi(t,x) \operatorname{div}_x(f(t,xV(t,x)) + \langle \nabla_x \varphi, f(t,x)V(t,x) \rangle \\ &= \partial_t(f(t,x)\varphi(t,x)) + \varphi(t,x) \operatorname{div}_x(f(t,xV(t,x)) + f(t,x) \langle \nabla_x \varphi, V(t,x) \rangle \\ &= \varphi(t,x)(\partial_t f(t,x) + \operatorname{div}_x(f(t,x)V(t,x))) + f(t,x(\partial_t \varphi(t,x) + f(t,x) \langle \nabla_x \varphi, V(t,x) \rangle. \end{aligned}$$

Combining this with the fact that the integral vanishes we obtain the following result:

$$\begin{aligned} &\int_{(0,T) \times \mathbb{R}^n} f(t,x)(\partial_t \varphi(t,x) + \langle V(t,x), \nabla_x \varphi(t,x) \rangle) dx dt \\ &= -\int_{(0,T) \times \mathbb{R}^n} \varphi(t,x)(\partial_t f(t,x) + \operatorname{div}_x(f(t,x) \cdot V(t,x)) dx dt \end{aligned}$$

So, if $\partial_t f + \operatorname{div}_x(fV) = 0$, then we have

$$\int_{(0,T) \times \mathbb{R}^n} f(t,x)(\partial_t \varphi(t,x) + \langle V(t,x), \nabla_x \varphi(t,x) \rangle) dx dt = 0,$$

as claimed. Conversely, if for any $\varphi \in C_c^1((0,T) \times \mathbb{R}^n)$ we have

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t \varphi(t,x) + V(t,x) \cdot \nabla_x \varphi(t,x)) f(t,x) dx dt = 0,$$

then $\partial_t f + \operatorname{div}_x(fV)$ must vanish almost-everywhere and thus by its continuity must vanish. This concludes the proof. ☐

C. Solutions

We will now see that weak solutions to the conservative transport equation exist under rather general circumstances and are given by a formula that is very similar to (and also compatible with) the case treated in last week's talk.

Theorem 4.10. *Let $V \in C^1([0,T] \times \mathbb{R}^n)$ be a vector field that satisfies (H1) and (H2) and let X be its characteristic flow. Further let $\mu_0 \in \mathcal{M}^+(\mathbb{R}^n)$. Then the Cauchy-problem*

$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mu V) = 0 \\ \mu|_{t_0} = \mu_0 \end{cases}$$

has a unique weak solution μ given by

$$\mu(t) = X(t,0,\cdot)\sharp\mu_0$$

for any $t \in [0,T]$.

Proof.
- We first show the existence of a weak solution. This is done by checking that the formula given in the theorem is indeed a solution. For this let $\varphi \in C_c^1((0,T) \times \mathbb{R}^n)$ be arbitrary. Then we define the function
$$t \mapsto \int_{\mathbb{R}^n} \varphi(t, X(t,0,y)) d\mu_0(y).$$
We know from the results on the flow that this is of class C^1. Now we observe that
$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} \varphi(t, X(t,0,y)) d\mu_0(y) \\ &= \int_{\mathbb{R}^n} \frac{d}{dt}\varphi(t, X(t,0,y)) d\mu_0(y) \\ &= \int_{\mathbb{R}^n} \partial_t\varphi(t, X(t,0,y)) + \langle \nabla_x \varphi(t, X(t,0,y)), \underbrace{\partial_t X(t,0,y)}_{=V(t,X(t,0,y))} \rangle d\mu_0(y) \\ &= \int_{\mathbb{R}^n} \partial_t \varphi(t,x) + \langle \nabla_x\varphi(t,x), V(t,x)\rangle d\mu(x), \end{aligned}$$
where we use the integral formula for the push-forward measure in the last line. Next we observe that
$$\int_{\mathbb{R}^n} \varphi(T, X(T,0,y)) d\mu_0(y) = 0$$
and
$$\int_{\mathbb{R}^n} \varphi(0, X(0,0,y)) d\mu_0(y) = 0$$
because φ has compact support in $(0,T) \times \mathbb{R}^n$. Thus
$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}\int_{\mathbb{R}^n} \varphi(t, X(t,0,y)) d\mu_0(y) \\ &= \int_0^T \int_{\mathbb{R}^n} \partial_t\varphi(t,x) + \langle \nabla_x\varphi(t,x), V(t,x)\rangle d\mu(x) dt, \end{aligned}$$
from which we conclude that μ is a weak solution to the conservative transport equation with the relevant initial data.
- What is left to show is the uniqueness of the solution. So let μ be any solution to the conservative transport equation with the relevant initial data. Further let $\psi \in C_c^1(\mathbb{R}^n)$ be arbitrary. Then set
$$\nu(t) = X(t,0,\cdot)\sharp\mu.$$
Now let $\xi \in C_c^\infty((0,T))$ be arbitrary. We define the C^1-function $\Psi : [0,T] \times \mathbb{R}^n$ as
$$t, y \mapsto \xi(t)\psi(X(0,t,y)).$$
We note that in the previous talk it was shown that there is a $\kappa > 0$ such that
$$|X(s,t,y)| \leq (|y| + \kappa T)e^{\kappa T}$$
for all $y \in \mathbb{R}^n$ and $s,t \in [0.T]$. Thus the fact that the support of ψ is compact implies as well that $\Psi(t,\cdot)$ has compact support for any $t \in [0,T]$. If we combine this with the fact that ξ has

compact support in $(0,T)$, we obtain that Ψ has compact support in $(0,T)\times\mathbb{R}^n$. Then we note that

$$\begin{aligned}&\partial_t\Psi(t,y)+\langle V(t,y),\nabla_y\Psi(t,y)\rangle\\ =&\xi(t)\langle\underbrace{\partial_t X(0,t,y)}_{=-V(t,y)},\nabla_y\psi\rangle+\xi'(t)\psi(y)+\langle V(t,y),\nabla_y\Psi(t,y)\rangle\\ =&\xi'(t)\psi(X(0,t,y)).\end{aligned}$$

We further note that

$$\begin{aligned}-\int_0^T\xi'(t)\left(\int_{\mathbb{R}^n}\psi(x)d\nu(t)(x)\right)dt=&-\int_0^T\int_{\mathbb{R}^n}\xi'(t)\psi(X(0,t,y))d\mu(t)(y)dt\\ =&-\int_0^T\int_{\mathbb{R}^n}\partial_t\Psi(t,y)+\langle V(t,y),\nabla_y\Psi(t,y)\rangle d\mu(t)(y)dt\\ =&0,\end{aligned}$$

where we use that μ is a weak solution to the transport equation in the last step. This implies that the weak derivative of the map

$$[0,T]\ni t\mapsto\int_{\mathbb{R}^n}\psi(x)d\nu(t)(x)\in\mathbb{R}$$

is 0, which in turn implies that this function is constant. Thus

$$\begin{aligned}\int_{\mathbb{R}^n}\psi(x)d\nu(t)(x)=&\int_{\mathbb{R}^n}\psi(x)d\nu(0)(x)\\ =&\int_{\mathbb{R}^n}\psi(x)d\mu(t)(x)=\int_{\mathbb{R}^n}\psi(x)d\mu_0(x)\end{aligned}$$

which implies (since $\psi\in C_c^1(\mathbb{R}^n)$ was arbitrary) that

$$\nu(t)=X(0,t,\cdot)\sharp\mu(t)=\mu_0\implies\mu(t)=X(t,0,\cdot)\sharp\mu_0$$

and thus concludes the proof.

□

We will now consider a simple example:

Example 4.11. Let $n=2$, $\mu_0=\delta_{(\ell,0)}$ for some $\ell\in\mathbb{R}$ and $V(t,(x_1,x_2))=(-x_2,x_1)$. Then we can see directly, that when we identify $\mathbb{R}^2\cong\mathbb{C}$ we get

$$X(t,s,y)=e^{i(s-t)}y.$$

And thus we obtain

$$\begin{aligned}\mu(t)=X(t,0,\cdot)\sharp\mu_0&=(y\mapsto e^{-it}y)\sharp\delta_{(\ell,0)}\\ &=\delta_{(\ell\cos t,\ell\sin t)}.\end{aligned}$$

Here it is useful to keep in mind why we even want to look at measure valued solutions: At some point we want to look at (physical) systems made up of many particles. The measure can then be interpreted as the probability distribution of a random choice of particle from this system. Our example would be the (boring) case of a system with just one particle.

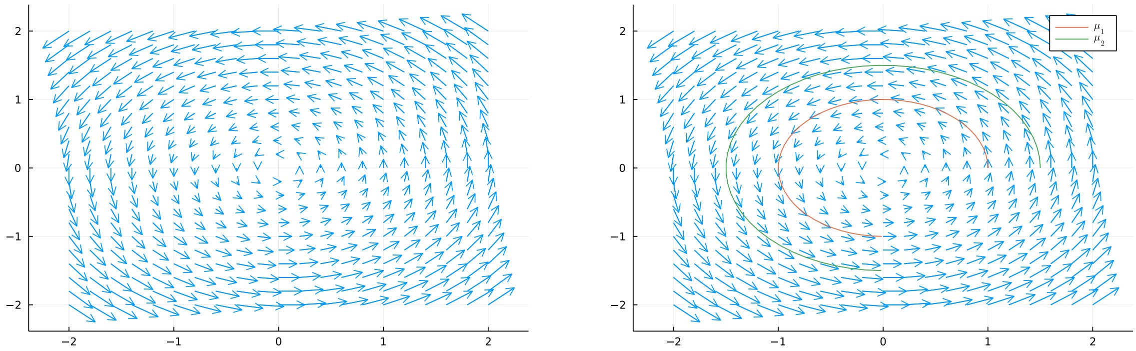
To conclude the talk we will see that if V additionally satisfies (H3) we get a C^1 solution for a C^1 initial value.

Theorem 4.12. *Let $V\in C^1([0,T]\times\mathbb{R}^n)$ be a vector field that satisfies (H1)-(H3) and let X be its characteristic flow. Further let $f_0\in C^1(\mathbb{R}^n)$. Then the Cauchy-problem*

$$\begin{cases}\partial_t f+\operatorname{div}_x(fV)=0\\ f|_{t_0}=f_0\end{cases}$$

has a unique weak solution μ given by

$$f(t,x)=f_0(X(0,t,x))\det(D_xX(0,t,x)).$$



(A) The vector field V. (B) V together with two solutions for $\ell = 1$ and $\ell = \frac{3}{2}$.

FIGURE 2. A figure visualizing example 4.11.

Proof.

- We first show that the solution to this equation is unique. For this it suffices to show that any solution with initial data $f_0 = 0$ vanishes. So assume $g \in C^1([0,T] \times \mathbb{R}^n)$ is such that
$$\begin{cases} \partial_t g + \operatorname{div}_x(gV) = 0 \\ g|_{t_0} = 0. \end{cases}$$
Then we further have
$$0 = \partial_t g + \operatorname{div}_x(gV) = \partial_t g + g \operatorname{div}_x V + g\langle V, \nabla_x g\rangle$$
$$\implies \partial_t g + g\langle V, \nabla_x g\rangle = -g \operatorname{div}_x V.$$
Together with the properties of the characteristic flow we conclude that the function
$$[0,T] \ni t \mapsto g(t, X(t,0,y))$$
is C^1 and satisfies for any $y \in \mathbb{R}^n$ the ordinary differential equation
$$\begin{cases} \frac{d}{dt} g(t, X(t,0,y)) = -g(t, X(t,0,y))(\operatorname{div}_x V)(t,0,y)) \\ g|_{t=0} = 0 \end{cases}$$
which means that it must vanish, which by $X(t,0,\cdot)$ being a diffeomorphism implies that g vanishes. Thus we have uniqueness.
- We also want to show that $f \in C^1([0,T] \times \mathbb{R}^n)$ and that our formula is correct. For this we decompose f_0 using the Japanese-Bracket[3]. We write
$$f_0 = \underbrace{\langle f_0 \rangle}_{=: f_0^{(1)}} - \underbrace{(-f_0 + \langle f_0 \rangle)}_{=: f_0^{(2)}}.$$
Now let
$$f^{(1)}(t,x) = f_0^{(1)}(X(0,t,x)) \det(D_x X(0,t,x))$$
$$f^{(2)}(t,x) = f_0^{(2)}(X(0,t,x)) \det(D_x X(0,t,x)).$$
Then note
$$\mu^{(1)}(t) := X(t,0,\cdot)\sharp(f_0^{(1)}\lambda) = f^{(1)}\lambda$$
$$\mu^{(2)}(t) := X(t,0,\cdot)\sharp(f_0^{(2)}\lambda) = f^{(2)}\lambda,$$
are solutions to the conservative transport equation with the respective initial data. Note that the problem has now the right form since both $f_0^{(1)}$ and $f_0^{(2)}$ are non-negative and thus $f_0^{(1)}\lambda, f_0^{(2)}\lambda \in \mathcal{M}^+(\mathbb{R}^n)$. Here we use our earlier result on the push-forward of the Lebesgue measure λ under a diffeomorphism. Now note that by linearity $\mu^{(1)} - \mu^{(2)}$ is a weak solution of the conservative transport equation with initial data $\mu_0 = f_0\lambda$. Thus $f = f^{(1)} - f^{(2)}$ since $f\lambda$ is then the unique (weak) solution to the conservative transport equation. However our formula gives us that f is C^1 and thus due to the result of the last talk it is a classical solution as well.

□

[3] $\langle x \rangle = \sqrt{1 + |x|^2}$.

References

[Gol13] F. Golse, Mean field kinetic equations, `https://metaphor.ethz.ch/x/2022/fs/401-4820-22L/notes/PolyKinetic.pdf`, 2013.