Seminar talk

Consider a system of particles (e.g. a gas) interacting with one another.
Consider further $\mathbb{R}^d$ to be the phase space of one individual particle (rather than the phase space of the entire system).

There are now different viewpoints:

(i) To describe the state of ~~th~~ ~~an entire~~ an ~~the~~ entire system of N particles we consider the individual ~~s~~ states ~~$z_1(t), \ldots, z_N(t) \in \mathbb{R}^N$~~ $z_1(t), \ldots, z_N(t) \in \mathbb{R}^d$ in phase space. ~~depending~~

(ii) Let $\mu_t \in P(\mathbb{R}^d)$ be a probability measure on the phase space $\mathbb{R}^d$. Then given that the system consists of N particles, ~~we~~ there are ~~(approximately?)~~ $N\mu(t)(A)$ particles with state in the region $A \subseteq \mathbb{R}^d$ at time t.

First some definitions:

Definition 1 (Interaction kernel)

We call a function $K: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfying

~~(HK1)~~ (i) $K(z,z') = -K(z',z)$ for all $z,z' \in \mathbb{R}^d$

i.e. K is skew symmetric

(ii) $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ and

$$\sup_{z' \in \mathbb{R}^d} |D_z K(z,z')| + \sup_{z \in \mathbb{R}^d} |D_{z'} (z,z')| \leq L$$

an interaction kernel.

Note the following consequence of the definition

Lemma 1 Let $K: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an interaction kernel then:

(i) $\sup_{z' \in \mathbb{R}^d} \| K(z_1, z') - K(z_2, z') \| \leq L \| z_1 - z_2 \|$

(ii) $\sup_{z \in \mathbb{R}^d} \| K(z, z_1) - K(z, z_2) \| \leq L \| z_1 - z_2 \|$

(iii) $\| K(z,z') \| \leq L (\|z\| + \|z'\|)$ for all $z,z' \in \mathbb{R}^d$.

For (iii) note that

$$\|K(z,z') - K(z',z')\| \leq \sup_{\cancel{z' \in \mathbb{R}^d},\ a \in \mathbb{R}^d} \|K(z,a) - K(z',a)\|$$

$$\leq L(\|z-z'\|) \leq L(\|z\| + \|z'\|).$$

Thus we concluded the proof $\square$.

Definition 2 (empirical measure) ~~The empirical measure of a N-tuple~~ $\cancel{Z_N \in (\mathbb{R}^d)^N}$ ~~is a Borel probability measure on~~ $\cancel{\mathbb{R}^d}$

Let $Z_N \in (\mathbb{R}^d)^N$, then with $Z_N = (z_1, \dots, z_N)$ define

$$\mu_{Z_N} = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j}.$$

This is a Borel probability measure, the so called <u>empirical measure</u>.

~~Definition~~ Lemma 3 Given an interaction kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, ~~Define the~~ The integral operator

$$\mathcal{K} : P_1(\mathbb{R}^d) \longrightarrow \cancel{\mathrm{Lip}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)}\ \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^d)$$

defined by $\mu \longmapsto (K\mu)(z) := \int_{\mathbb{R}^d} K(z,z')\, d\mu(z')$

is well defined.

Pf.
Follows directly from Lemma 1. $\square$

Now let us return to the two different perspectives introduced in the beginning:

In view point (i) we described the state of our system of N particles by the individual states $z_1(t), \dots, z_N(t) \in \mathbb{R}^d$.
Now let $K: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an interaction kernel, then we denote by $K(z_i, z_j)$ the interaction of the ~~ith and jth particle.~~ the jth on the ith particle and encode the dynamics of our system into the ~~Systeo~~ system of ODEs

$$\frac{dz_i}{dt}(t) = \sum_{\substack{j=1 \\ j \neq i}}^{N} K(z_i, z_j) \quad i,j = 1, \dots, N.$$

Since K is skew symmetric $K(z,z) = 0$ for all $z \in \mathbb{R}^d$. ~~and~~
Rescalling the time variable and using this observation we can therefore rewrite the system into the following form:

$$\dot{z}_i(t) = \frac{1}{N}\sum_{j=1}^{N} K(z_i, z_j) \qquad i = 1, \dots, N. \quad (1)$$

Clearly the empirical ~~probability~~ measure $\mu_{z_N} = \sum_{j=1}^{N} \delta_{z_j}$ where $z_N = (z_1, \dots, z_N) \in (\mathbb{R}^d)^N$ is precisely the measure we sought in viewpoint (ii).

It turns out that the dynamics of the system in this viewpoint is controlled by the mean field PDE

$$\partial_t \mu + \mathrm{div}_z(\mu K\mu) = 0 \qquad (2)$$

introduced in last weeks talk.

We can ~~show~~ prove that $z_N = (z_1(t), \dots, z_n(t))$ solves (1) if and only if μ_{z_N} solves (2); the first of these directions we will be able to show today:

Theorem 1 Let $K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an interaction kernel as defined above. Then

(i) For each $N \geq 1$ and each N-tuple $z_N^{in} = (z_1^{in}, \dots, z_n^{in})$

$$(3) \quad \begin{cases} \dot{z}_i(t) = \frac{1}{N}\sum_{j=1}^{N} K(z_i(t), z_j(t)) & i = 1, \dots, N \\ z_i(0) = z_i^{in} \end{cases}$$

has a unique solution of class C^1 on $\mathbb{R}$.

$$t \longmapsto z_N(t) = (z_1(t), \dots, z_N(t)).$$

(ii) The empirical measure $\mu_{z_N(t)}$ is a weak solution of

$$(4) \quad \begin{cases} \partial_t \mu + \mathrm{div}_z(\mu \mathcal{K}\mu) = 0 \\ \mu|_{t=0} = \mu^{in} := \mu_{z_N^{in}}. \end{cases}$$

Before we continue with the proof, recall:

<u>Theorem 2</u>:

Assume $V \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$,

$D_x V \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^{N \times N})$

and $\forall t \in [0,T]$ $\forall x \in \mathbb{R}^N$ $|V(t,x)| \le K(1+|x|)$ for some $K \in \mathbb{R}$. Then for all $t \in [0,T]$ and $x \in \mathbb{R}^N$ there exists a unique characteristic defined on $[0,T]$ and satisfying $X(t) = x$. We denote it by $X(s,t,x)$ and we have $X \in C^1([0,T]_s \times [0,T]_t \times \mathbb{R}_x^N)$ and $\partial_s D_x X$ and $D_x \partial_s X$ exist and are equal in $C([0,T]_s \times [0,T]_t \times \mathbb{R}_x^N)$. ~~If moreover~~

Theorem 3 (Thm. 4.10 in last weeks talk)

Let $V \in C^1([0,T] \times \mathbb{R}^N)$ be a vector field that satisfies the conditions of theorem 2 and let X be its characteristic flow. Further let $\mu_0 \in \mathcal{M}^+(\mathbb{R}^N)$. Then the Cauchy problem

$$\begin{cases} \partial_t \mu + \operatorname{div}_z(\mu V) = 0 \\ \mu|_{t=0} = \mu_0 \end{cases}$$

has a unique weak solution

$$\mu(t) = X(t,0,\cdot) \# \mu_0$$

Proof of Thm 1:

See next page

Proof of theorem 1:

(i) $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}$, then

$$\dot{Z} = \frac{1}{N}\begin{pmatrix} \sum_{j=1}^{N} K(z_1, z_j) \\ \vdots \\ \sum_{j=1}^{N} K(z_N, z_j) \end{pmatrix} = \frac{1}{N}\sum_{j=1}^{N}\begin{pmatrix} K(z_1, z_j) \\ \vdots \\ K(z_N, z_j) \end{pmatrix}$$

$$=: a(t, Z)$$

where $\|a(t,Z)\|_1 \leq \frac{1}{N}\sum_{j=1}^{N}\left\|\begin{pmatrix} K(z_1, z_j) \\ \vdots \\ K(z_N, z_j) \end{pmatrix}\right\|_1$

$$= \frac{1}{N}\sum_{i,j=1}^{N}\underbrace{\|K(z_i, z_j)\|}_{\leq \cancel{\|z_i - z_j\|}\, L(\|z_i\|_1 + \|z_j\|_1)\ (\text{Lemma 1(ii)})}$$

$$\leq 2L\|Z\| \leq 2L(1 + \|Z\|)$$

$\overset{\text{Theorem 2}}{\Longrightarrow}$ There exists a unique solution of class C^1 as required.

(ii)

Given the solution $z_N(t) = (z_1(t), \dots, z_n(t))$ to (3) define

$$V(t,z) := \int_{\mathbb{R}^d} K(z,z')\, d\mu_{z_N(t)} = \frac{1}{N}\sum_{j=1}^{N} K(z, z_j)$$

Thus $V \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$ and $D_x V \in C([0,T]\times\mathbb{R}^N, \mathbb{R}^{N\times N})$

and finally $$\|V(t,x)\| \leq \int_{\mathbb{R}^d} \|K(z,z')\|\, d\mu_{z_N(t)}$$

$$\leq L \int_{\mathbb{R}^d} \|z\| + \|z'\|\, d\mu_{z_N(t)} = L\|z\| + L\cdot\alpha$$

$$\leq \max\{L, L\alpha\}(1 + \|z\|).$$

Thus by Thm 3 there exists a unique solution of

$$\begin{cases} \partial_t \mu + \operatorname{div}_z(\mu V) = 0 \\ \mu|_{t=0} = \mu|_{z_N^{in}}. \end{cases}$$

which can be given by

$$\mu(t) = X(t,0,\cdot)\#\mu_{Z_N^{in}}$$

$$\mu(t)(\{z\})\ \cancel{\mu(\{z\})} = \mu_{Z_N^{in}}\left(X(t,0,\cdot)^{-1}(\{z\})\right) = 1$$

if and only if $z \in \mathbb{R}^d$ is a component of $Z_N^{in} \in (\mathbb{R}^d)^N$

Thus $\mu(t) = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j(t)} = \mu_{Z_N(t)}$

and so $\partial_t \mu_{Z_N} + \operatorname{div}(\mu_{Z_N} V) = 0$

$$\Rightarrow \partial_t \mu_{Z_N} + \operatorname{div}_z(\mu_{Z_N} K * \mu_{Z_N}) = 0. \quad \square$$

We only used the exsistence part and the concrete formula from Theorem 3.