Theorem 3.2.2

Assume the interaction kernel $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies

(HK1) $\quad K(z,z') = -K(z',z) \quad \forall z, z' \in \mathbb{R}^d$

(HK2) $\quad \exists L \geq 0 : \quad \sup_{z' \in \mathbb{R}^d} |\nabla_z K(z,z')| \leq L \quad \& \quad \sup_{z \in \mathbb{R}^d} |\nabla_{z'} K(z,z')| \leq L.$

Consider the problem:

$$\begin{cases} \partial_t Z(t, \zeta^{in}, \mu^{in}) = (K\mu(t))(Z(t, \zeta^{in}, \mu^{in})), \\ \mu(t) = Z(t, \cdot, \mu^{in}) \# \mu^{in}, \\ Z(0, \zeta^{in}, \mu^{in}) = \zeta^{in}. \end{cases}$$

Then for each $\zeta^{in} \in \mathbb{R}^d$ & each Borel probability measure $\mu^{in} \in \mathcal{P}_1(\mathbb{R}^d)$

$\exists!$ solution of the problem, which is denoted by

$$\mathbb{R} \ni t \mapsto Z(t, \zeta^{in}, \mu^{in}) \in \mathbb{R}^d$$

& is of class C^1.

Rem.: • Here: single-particle phase space $\mathbb{R}^d$

Last time: N-particle phase space

• This ODE is precisely the equation of characteristics for the mean field PDE.

So we call this ODE the equations of "mean field characteristics",

& its solution Z the "mean field characteristic flow".

Proposition 3.2.3 (Relation between Z & flow T_t)

associated to the N-particle ODE system

Assume the interaction kernel $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies (HK1 & HK2).

For each $Z_N^{in} = (z_1^{in}, \dots, z_N^{in})$, the solution $T_t Z_N^{in} = (z_1(t), \dots, z_N(t))$ of the N-body problem & the mean field characteristic flow $Z(t, \zeta^{in}, \mu^{in})$ satisfy

$$z_i(t) = Z(t, z_i^{in}, \mu_{Z_N^{in}}), \quad i = 1, \dots, N \quad \forall t \in \mathbb{R}.$$

Proof of Prop. 3.2.3

Define $\zeta_i(t) := Z(t, z_i^{in}, \mu_{Z_N^{in}}), \quad i = 1, \dots, N.$

Then $\mu(t) = Z(t, \cdot, \mu_{Z_N^{in}}) \# \mu_{Z_N^{in}} \overset{(*)}{=} \frac{1}{N} \sum_{j=1}^N \delta_{\zeta_j(t)} \quad \forall t \in \mathbb{R}.$

$\Rightarrow \dot{\zeta}_i(t) = (K\mu(t))(\zeta_i(t)) \overset{(**)}{=} \frac{1}{N} \sum_{j=1}^N K(\zeta_i(t), \zeta_j(t)), \quad i = 1, \dots, N \quad \forall t \in \mathbb{R},$

& $\zeta_i(0) = Z(0, z_i^{in}, \mu^{in}) = z_i^{in}, \quad i = 1, \dots, N.$

By Thm. 3.2.1 (uniqueness of the solution of the N-particle equation)

we have $\zeta_i(t) = z_i(t) \quad \forall i = 1, \dots, N \quad \forall t \in \mathbb{R}.$

(*) Def. 3.1.1: $\forall$ N-tuple $Z_N(z_1, \dots, z_N) \in \mathbb{R}^{dN}$, empirical measure is $\mu_{Z_N} := \frac{1}{N} \sum_{j=1}^N \delta_{z_j}$.

(**) $Kf(t,z) := \int_{\mathbb{R}^d} K(z,z') f(t, dz')$

Proof of Thm. 3.2.2 (simple variant of the proof of the Cauchy-Lipschitz thm.) 2

Let $\mu^{in} \in \mathcal{P}_1(\mathbb{R}^d)$ & $C_1 := \int_{\mathbb{R}^d} |z| \mu^{in}(dz)$.

Let $X := \left\{ v \in C(\mathbb{R}^d; \mathbb{R}^d) : \sup_{z \in \mathbb{R}^d} \frac{|v(z)|}{1+|z|} < \infty \right\}$.

X is a Banach space for the norm $\|v\|_X := \sup_{z \in \mathbb{R}^d} \frac{|v(z)|}{1+|z|}$.

(HK2) $\Rightarrow \forall v, w \in X$: $\left| \int_{\mathbb{R}^d} K(v(z), v(z')) \mu^{in}(dz') - \int_{\mathbb{R}^d} K(w(z), w(z')) \mu^{in}(dz') \right|$

$$\leq L \int_{\mathbb{R}^d} (|v(z) - w(z)| + |v(z') - w(z')|) \mu^{in}(dz')$$

$$\leq L \|v-w\|_X (1+|z|) + L \|v-w\|_X \int_{\mathbb{R}^d} (1+|z'|) \mu^{in}(dz')$$

$$= L \|v-w\|_X (1+|z|+1+C_1)$$

$$\leq L \|v-w\|_X (2+C_1)(1+|z|) \qquad (1)$$

Define sequence $(Z_n)_{n \geq 0}$ as:

$$\begin{cases} Z_{n+1}(t,\zeta) = \zeta + \int_0^t \int_{\mathbb{R}^d} K(Z_n(t,\zeta), Z_n(t,\zeta')) \mu^{in}(d\zeta') \, ds, & n \geq 0 \\ Z_0(t,\zeta) = \zeta . \end{cases}$$

By induction & by (1):

$$\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X \overset{\star}{\leq} \frac{((2+C_1)L|t|)^n}{n!} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X .$$

Since $|Z_1(t,\zeta) - Z_0(t,\zeta)| = |Z_1(t,\zeta) - \zeta| = \left| \int_0^t \int_{\mathbb{R}^d} K(\zeta,\zeta') \mu^{in}(d\zeta') \, ds \right|$

$$\leq \int_0^{|t|} \int_{\mathbb{R}^d} L(|\zeta| + |\zeta'|) \mu^{in}(d\zeta') \, ds$$

$$= \int_0^{|t|} L(|\zeta| + C_1) \, ds$$

$$\leq L(1+C_1)(1+|\zeta|)|t| \qquad (1.5)$$

we have $\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X \overset{\star\star}{\leq} \frac{((2+C_1)L|t|)^{n+1}}{n!}$.

So $\forall T > 0$ $Z_n(t,\cdot) \longrightarrow Z(t,\cdot)$ in X uniformly on $[-T,T]$, where $Z \in C(\mathbb{R}, X)$ is

(2) $$Z(t,\zeta) = \zeta + \int_0^t \int_{\mathbb{R}^d} K(Z(s,\zeta), Z(s,\zeta')) \mu^{in}(d\zeta') \, ds \qquad \forall t \in \mathbb{R}, \ \forall \zeta \in \mathbb{R}^d.$$

$\star$ see Appendix $\quad$ $\star\star$ see Appendix

Uniqueness of Z:

Assume $Z, \tilde{Z} \in C(\mathbb{R}; X)$ both satisfy ②, then

$$Z(t,\zeta) - \tilde{Z}(t,\zeta) = \int_{\mathbb{R}^d} \left(K(Z(s,\zeta), Z(s,\zeta')) - K(\tilde{Z}(s,\zeta), \tilde{Z}(s,\zeta'))\right)\mu^{in}(d\zeta')$$

$\Rightarrow \forall t \in \mathbb{R}$: $\|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X \leq L(2+C_1)\left|\int_0^t \|Z(s,\cdot) - \tilde{Z}(s,\cdot)\|_X \, ds\right|$.

$\Rightarrow \|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X = 0$ (by Grönwall's inequality) ⊛

$\Rightarrow Z = \tilde{Z}$

Hence, ② has only one solution $Z \in C(\mathbb{R}; X)$.

Continuity of $s \mapsto \int_{\mathbb{R}^d} K(Z(s,\zeta), Z(s,\zeta'))\mu^{in}(d\zeta')$ on $\mathbb{R}$ follows from the fact that $Z \in C(\mathbb{R}_+; X)$, $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies (HK2) & $\mu^{in} \in \mathcal{P}_1(\mathbb{R}^d)$.

Using this & ②, we have that $t \mapsto Z(t,\zeta)$ is of class C^1 on $\mathbb{R}$ & satisfies

$$\begin{cases} \partial_t Z(t,\zeta) = \int_{\mathbb{R}^d} K(Z(t,\zeta), Z(t,\zeta'))\mu^{in}(d\zeta') \\ Z(0,\zeta) = \zeta \end{cases}.$$

Substituting $z' = Z(t,\zeta')$ we get $\int_{\mathbb{R}^d} K(Z(t,\zeta), Z(t,\zeta'))\mu^{in}(d\zeta') = \int_{\mathbb{R}^d} K(Z(t,\zeta), z') Z(t,\cdot)\#\mu^{in}(dz')$, so that $Z \in C(\mathbb{R}; X)$ is the unique solution of the mean field characteristic equation.

■

⊛ Grönwall's inequality: If $v(t) \leq b + a\int_0^t v(s)\,ds$ $\forall t \in [0,T]$, then $v(t) \leq e^{at} b$ $\forall t \in [0,T]$.
Here $L(2+C_1)\int_0^t \|Z(s,\cdot) - \tilde{Z}(s,\cdot)\|_X \, ds = b + a\int_0^t v(s)\,ds$, i.e. $b = 0$ in our case
therefore $v(t) \leq e^{at} \cdot 0 = 0$ and so $\|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X = v(t) \leq 0$.
Since $\|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X \geq 0$, we have equality.

3.3 Dobrushin's stability estimate & the mean field limit

3.3.1 The Monge-Kantorovich distance

Def.: • $\forall r \geq 1$ denote by $\mathcal{P}_r(\mathbb{R}^d)$ the set of Borel probability measure on $\mathbb{R}^d$ with a finite moment of order r, i.e. satisfying $\int_{\mathbb{R}^d} |z|^r \mu(dz) < \infty$.

• Given $\mu, \nu \in \mathcal{P}_r(\mathbb{R}^d)$, we define $\Pi(\mu,\nu)$ to be the set of Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$, i.e. $\forall \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$

$$\pi \in \Pi(\mu,\nu) \Leftrightarrow \iint_{\mathbb{R}^d\times\mathbb{R}^d} (\phi(x) + \psi(y))\, \pi(dxdy) = \int_{\mathbb{R}^d} \phi(x)\mu(dx) + \int_{\mathbb{R}^d} \psi(y)\nu(dy)$$

$$\forall \phi, \psi \in C(\mathbb{R}^d) \text{ s.t. } \phi(z) = O(|z|^r) \ \& \ \psi(z) = O(|z|^r) \text{ as } |z| \to \infty.$$

Rem.: • $\pi \in \Pi(\mu,\nu)$ is sometimes called "couplings of μ & ν".

• $\mu, \nu \in \mathcal{P}_r(\mathbb{R}^d)$ for $r > 0$, then $\Pi(\mu,\nu) \subset \mathcal{P}_r(\mathbb{R}^d \times \mathbb{R}^d)$.

Definition 3.3.1

$\forall r \geq 1$, $\forall \mu, \nu \in \mathcal{P}_r(\mathbb{R}^d)$, the Monge-Kantorovich distance $\mathrm{dist}_{MK,r}(\mu,\nu)$ between μ & ν is defined by the formula:

$$\mathrm{dist}_{MK,r}(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^r \, \pi(dxdy) \right)^{\frac{1}{r}}$$

Rem.: We use $\mathrm{dist}_{MK,r}(\mu,\nu)$ as a convenient tool for studying the stability of the mean field characteristic flow.

Proposition 3.3.2 (special case $r=1$)

In the case where $r=1$, the Monge-Kantorovich distance is also given by

$$\mathrm{dist}_{MK,1}(\mu,\nu) = \sup_{\substack{\phi \in \mathrm{Lip}(\mathbb{R}^d) \\ \mathrm{Lip}(\phi) \leq 1}} \left| \int_{\mathbb{R}^d} \phi(z)\mu(dz) - \int_{\mathbb{R}^d} \phi(z)\nu(dz) \right|,$$

where $\mathrm{Lip}(\phi) := \sup_{x \neq y \in \mathbb{R}^d} \dfrac{|\phi(x) - \phi(y)|}{|x-y|}$ is the Lipschitz constant of ϕ.

3.3.2 Dobrushin's estimate

Rem.: • From Prop. 3.2.3 we know that the mean field characteristic flow Z contains all relevant information about both the mean field PDE & the N-particle ODE system.

• Dobrushin's approach to the mean field limit is based on the idea of proving the stability of the mean field characteristic flow $Z(t, \zeta^{in}, \mu^{in})$ in both the initial position in phase space ζ^{in} & the initial distribution μ^{in}.

• We see that the Monge-Kantorovich distance is the best adapted mathematical tool to measure this stability.

Idea: (rests on the following key computation)

Let $\zeta_1^{in}, \zeta_2^{in} \in \mathbb{R}^d$ & let $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbb{R}^d)$. Then

$$\begin{aligned} &Z(t, \zeta_1, \mu_1^{in}) - Z(t, \zeta_2, \mu_2^{in}) \\ &= \zeta_1 - \zeta_2 + \int_0^t \int_{\mathbb{R}^d} K(Z(s, \zeta_1, \mu_1^{in}), z') \mu_1(s, dz') ds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} K(Z(s, \zeta_2, \mu_2^{in}), z') \mu_2(s, dz') ds. \end{aligned} \tag{3}$$

Using $\mu_j(t) = Z(t, \cdot, \mu_j^{in}) \# \mu_j^{in}$ for $j = 1, 2$, we have

$$\int_{\mathbb{R}^d} K(Z(s, \zeta_j, \mu_j^{in}), z') \mu_j(s, dz') = \int_{\mathbb{R}^d} K(Z(s, \zeta_j, \mu_j^{in}), Z(s, \zeta_j', \mu_j^{in})) \mu_j^{in}(d\zeta_j').$$

So for each coupling $\pi^{in} \in \mathcal{P}_1(\mu_1^{in}, \mu_2^{in})$ one has

$$\begin{aligned} &\int_{\mathbb{R}^d} K(Z(s, \zeta_1, \mu_1^{in}), Z(s, \zeta_1', \mu_1^{in})) \mu_1^{in}(d\zeta_1') - \int_{\mathbb{R}^d} K(Z(s, \zeta_2, \mu_2^{in}), Z(s, \zeta_2', \mu_2^{in})) (d\zeta_2') \\ &\overset{*}{=} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(K(Z(s, \zeta_1, \mu_1^{in}), Z(s, \zeta_1', \mu_1^{in})) - K(Z(s, \zeta_2, \mu_2^{in}), Z(s, \zeta_2', \mu_2^{in})) \right) \pi^{in}(d\zeta_1', d\zeta_2'). \end{aligned}$$

Plugging in into (3) we get:

$$\begin{aligned} &Z(t, \zeta_1, \mu_1^{in}) - Z(t, \zeta_2, \mu_2^{in}) \\ &= \zeta_1 - \zeta_2 + \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(K(Z(s, \zeta_1, \mu_1^{in}), Z(s, \zeta_1', \mu_1^{in})) - K(Z(s, \zeta_2, \mu_2^{in}), Z(s, \zeta_2', \mu_2^{in})) \right) \pi(d\zeta_1', d\zeta_2') ds. \end{aligned} \tag{4}$$

The last equality is the key observation in Dobrushin's argument, which explains why it is natural to use the Monge-Kantorovich distance.

* by def. of coupling measure

By (HK2) on the interaction kernel K we have $\forall a, a', b, b' \in \mathbb{R}^d$

$$|K(a,a') - K(b,b')| \leq |K(a,a') - K(b,a')| + |K(b,a') - K(b,b')| \qquad (5)$$
$$\leq L|a-b| + L|a'-b'|.$$

(4) $\Rightarrow |Z(t,\xi_1,\mu_1^{in}) - Z(t,\xi_2,\mu_2^{in})|$

$$\overset{(***)}{\leq} |\xi_1 - \xi_2| + L\int_0^t |Z(s,\xi_1,\mu_1^{in}) - Z(s,\xi_2,\mu_2^{in})|\, ds + L\underbrace{\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |Z(s,\xi_1',\mu_1^{in}) - Z(s,\xi_2',\mu_2^{in})|\, \pi^{in}(d\xi_1', d\xi_2')\, ds}_{=: D[\pi^{in}](s)}$$

Integrating both sides of the inequality w.r.t. $\pi^{in}(d\xi_1, d\xi_2)$ leads to

$$D[\pi^{in}](t) \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi_1 - \xi_2|\, \pi^{in}(d\xi_1, d\xi_2) + 2L\int_0^t D[\pi^{in}](s)\, ds$$
$$= D[\pi^{in}](0) + 2L\int_0^t D[\pi^{in}](s)\, ds.$$

Gronwall's inequality $\Rightarrow \forall t \in \mathbb{R}$: $D[\pi^{in}](t) \leq D[\pi^{in}](0)e^{2L|t|}$.

Theorem 3.3.3 (Dobrushin)

Assume $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies (HK1-HK2). Let $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbb{R}^d)$. $\forall t \in \mathbb{R}$, let

$$\begin{cases} \mu_1(t) = Z(t,\cdot,\mu_1^{in}) \# \mu_1^{in} \\ \mu_2(t) = Z(t,\cdot,\mu_2^{in}) \# \mu_2^{in} \end{cases}$$

where Z is the mean field characteristic flow defined in Thm. 3.2.2.
Then $\forall t \in \mathbb{R}$: $\operatorname{dist}_{MK,1}(\mu_1(t), \mu_2(t)) \leq e^{2L|t|} \operatorname{dist}_{MK,1}(\mu_1^{in}, \mu_2^{in})$.

Proof: $\forall \mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbb{R}^d)$ & $\forall \pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})$: $D[\pi^{in}](t) \leq D[\pi^{in}](0)e^{2L|t|}$ $\forall t \in \mathbb{R}$.
Since $Z(t,\cdot,\mu_j^{in}) \# \mu_j^{in} = \mu_j(t)$ for $j = 1,2$
the map $\Phi_t : (\xi_1, \xi_2) \mapsto (Z(t,\xi_1,\mu_1^{in}), Z(t,\xi_2,\mu_2^{in}))$ satisfies
$\Phi_t \# \pi^{in} = \pi^{in} \in \Pi(\mu_1(t), \mu_2(t))$ $\forall t \in \mathbb{R}$, since $\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})$.
Thus,

$$\operatorname{dist}_{MK,1}(\mu_1(t), \mu_2(t)) = \inf_{\pi \in \Pi(\mu_1(t),\mu_2(t))} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi_1 - \xi_2|\, \pi(d\xi_1, d\xi_2)$$
$$\leq \inf_{\pi^{in} \in \Pi(\mu_1^{in},\mu_2^{in})} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |Z(t,\xi_1,\mu_1^{in}) - Z(t,\xi_2,\mu_2^{in})|\, \pi^{in}(d\xi_1, d\xi_2)$$
$$= \inf_{\pi^{in} \in \Pi(\mu_1^{in},\mu_2^{in})} D[\pi^{in}](t)$$
$$\leq e^{2L|t|} \inf_{\pi^{in} \in \Pi(\mu_1^{in},\mu_2^{in})} D[\pi^{in}](0)$$
$$= e^{2L|t|} \operatorname{dist}_{MK,1}(\mu_1^{in}, \mu_2^{in}) \qquad \blacksquare$$

Appendix

$$\circledast \quad \|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X = \sup_{z\in\mathbb{R}^d} \frac{|Z_{n+1}(t,z) - Z_n(t,z)|}{1+|z|} \le \frac{((2+C_1)L|t|)^n}{n!} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X$$

Proof by induction:

for $n=0$: $\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X = \underbrace{\frac{((2+C_1)L|t|)^0}{0!}}_{=1} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X$

for $n=1$:

$$\|Z_2(t,\cdot) - Z_1(t,\cdot)\|_X = \sup_{z\in\mathbb{R}^d} \frac{|Z_2(t,z) - Z_1(t,z)|}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} \frac{\left|z + \int_0^t\int_{\mathbb{R}^d} K(Z_1(s,z), Z_1(s,z'))\,\mu^{in}(dz')\,ds - \left(z + \int_0^t\int_{\mathbb{R}^d} K(Z_0(s,z), Z_0(s,z'))\,\mu^{in}(dz')\,ds\right)\right|}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} \frac{\left|\int_0^t\int_{\mathbb{R}^d} K(Z_1(s,z), Z_1(s,z'))\,\mu^{in}(dz') - \int_{\mathbb{R}^d} K(Z_0(s,z), Z_0(s,z'))\,\mu^{in}(dz')\,ds\right|}{1+|z|}$$

$$\le \sup_{z\in\mathbb{R}^d} \frac{\int_0^t \left|\int_{\mathbb{R}^d} K(Z_1(s,z), Z_1(s,z'))\,\mu^{in}(dz') - \int_{\mathbb{R}^d} K(Z_0(s,z), Z_0(s,z'))\,\mu^{in}(dz')\right| ds}{1+|z|}$$

$$\overset{1}{\le} \sup_{z\in\mathbb{R}^d} \frac{\int_0^{|t|} L\|Z_1(s,\cdot) - Z_0(s,\cdot)\|_X\,(2+C_1)(1+|z|)\,ds}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} L(2+C_1)\int_0^t \|Z_1(s,\cdot) - Z_0(s,\cdot)\|_X\,ds$$

$$= \sup_{z\in\mathbb{R}^d} L(2+C_1)\int_0^t \sup_{\zeta\in\mathbb{R}^d} \frac{|Z_1(s,\zeta) - Z_0(s,\zeta)|}{1+|\zeta|}\,ds$$

$$= \sup_{z\in\mathbb{R}^d} L(2+C_1)\int_0^t \sup_{\zeta\in\mathbb{R}^d} \frac{\left|\zeta + \int_0^s\int_{\mathbb{R}^d} K(Z_0(u,\zeta), Z_0(u,\zeta'))\,\mu^{in}(d\zeta')\,du - \zeta\right|}{1+|\zeta|}\,ds$$

$$\overset{s<t}{\le} \sup_{z\in\mathbb{R}^d} L(2+C_1)\int_0^t \sup_{\zeta\in\mathbb{R}^d} \frac{\left|\int_0^t\int_{\mathbb{R}^d} K(Z_0(u,\zeta), Z_0(u,\zeta'))\,\mu^{in}(d\zeta')\,du\right|}{1+|\zeta|}\,ds$$

$$= \sup_{z\in\mathbb{R}^d} L(2+C_1)\int_0^t \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X\,ds$$

$$= \sup_{z\in\mathbb{R}^d} L(2+C_1)\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X \int_0^t ds$$

$$\le L(2+C_1)\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X\,|t|$$

$$= \frac{((2+C_1)L|t|)^1}{1!}\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X$$

Assume it holds for $n < i$, then for $n = i$:

$$\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X = \sup_{z\in\mathbb{R}^d} \frac{|Z_{n+1}(t,z) - Z_n(t,z)|}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} \frac{\left| z + \int_0^t \int_{\mathbb{R}^d} K(Z_n(s,z), Z_n(s,z'))\,\mu^{in}(dz')\,ds - \left(z + \int_0^t\int_{\mathbb{R}^d} K(Z_{n-1}(s,z), Z_{n-1}(s,z'))\,\mu^{in}(dz')\,ds\right)\right|}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} \frac{\left| \int_0^t \left(\int_{\mathbb{R}^d} K(Z_n(s,z), Z_n(s,z'))\,\mu^{in}(dz') - \int_{\mathbb{R}^d} K(Z_{n-1}(s,z), Z_{n-1}(s,z'))\,\mu^{in}(dz')\right) ds\right|}{1+|z|}$$

$$\leq \sup_{z\in\mathbb{R}^d} \frac{\int_0^t \left| \int_{\mathbb{R}^d} K(Z_n(s,z), Z_n(s,z'))\,\mu^{in}(dz') - \int_{\mathbb{R}^d} K(Z_{n-1}(s,z), Z_{n-1}(s,z'))\,\mu^{in}(dz')\right| ds}{1+|z|}$$

$$\overset{1}{\leq} \sup_{z\in\mathbb{R}^d} \frac{\int_0^t L\,\|Z_n(s,\cdot) - Z_{n-1}(s,\cdot)\|_X\,(2+C_1)(1+|z|)\,ds}{1+|z|}$$

$$= \sup_{z\in\mathbb{R}^d} \frac{L(2+C_1)\cancel{(1+|z|)} \int_0^t \|Z_n(s,\cdot) - Z_{n-1}(s,\cdot)\|_X\,ds}{\cancel{1+|z|}}$$

$$\overset{\text{I.S.}}{\leq} L(2+C_1) \int_0^t \frac{((2+C_1)L|s|)^n}{n!} \|Z_1(s,\cdot) - Z_0(s,\cdot)\|_X\,ds$$

$$\overset{s<t}{\leq} \frac{(L(2+C_1))^{n+1}}{n!} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X \int_0^t |s|^n\,ds$$

$$= \frac{(L(2+C_1))^{n+1}}{n!} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X \frac{1}{n+1}|t|^{n+1}$$

$$= \frac{(L(2+C_1)|t|)^{n+1}}{(n+1)!} \|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X$$

$\blacksquare$

$$(\star\star)\quad \|Z_1(t,\cdot)-Z_0(t,\cdot)\|_X = \sup_{\xi\in\mathbb{R}^d} \frac{|Z_1(t,\xi)-Z_0(t,\xi)|}{1+|\xi|} \overset{(1.5)}{\leq} \sup_{\xi\in\mathbb{R}^d} \frac{L\,(1+C_1)\,(1+|\xi|)\,|t|}{1+|\xi|}$$

This implies that:

$$\|Z_{n+1}(t,\cdot)-Z_n(t,\cdot)\|_X \overset{(\star)}{\leq} \frac{((2+C_1)L\,|t|)^n}{n!}\,\|Z_1(t,\cdot)-Z_0(t,\cdot)\|_X$$

$$\overset{(\star\star)}{\leq} \frac{((2+C_1)L\,|t|)^n}{n!}\,\sup_{\xi\in\mathbb{R}^d} \frac{L(1+C_1)(1+|\xi|)\,|t|}{1+|\xi|}$$

$$\leq \sup_{\xi\in\mathbb{R}^d} \frac{((2+C_1)L\,|t|)^n}{n!}\cdot\frac{L(1+C_1)\,\cancel{(1+|\xi|)}\,|t|}{\cancel{1+|\xi|}}$$

$$= \frac{((2+C_1)L\,|t|)^n}{n!}\cdot L\,(1+C_1)\,|t|$$

$$\leq \frac{((2+C_1)L\,|t|)^n}{n!}\cdot L\,(2+C_1)\,|t|$$

$$= \frac{((2+C_1)L\,|t|)^{n+1}}{n!}$$

$\star\star\star$ $|Z(t,\zeta_1,\mu_1^{in}) - Z(t,\zeta_2,\mu_2^{in})|$

$$= \left|\zeta_1 - \zeta_2 + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(K(Z(s,\zeta_1,\mu_1^{in}), Z(s,\zeta_1',\mu_1^{in})) - K(Z(s,\zeta_2,\mu_2^{in}), Z(s,\zeta_2',\mu_2^{in}))\right) \pi^{in}(d\zeta_1', d\zeta_2')\, ds\right|$$

$$\leq |\zeta_1 - \zeta_2| + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left|K(Z(s,\zeta_1,\mu_1^{in}), Z(s,\zeta_1',\mu_1^{in})) - K(Z(s,\zeta_2,\mu_2^{in}), Z(s,\zeta_2',\mu_2^{in}))\right| \pi^{in}(d\zeta_1', d\zeta_2')\, ds$$

$$\overset{(5)}{\leq} |\zeta_1 - \zeta_2| + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \underbrace{L\,|Z(s,\zeta_1,\mu_1^{in}) - Z(s,\zeta_2,\mu_2^{in})|}_{\text{doesn't depend on } \zeta_1' \,\&\, \zeta_2'} + L\,|Z(s,\zeta_1',\mu_1^{in}) - Z(s,\zeta_2',\mu_2^{in})|\ \pi^{in}(d\zeta_1', d\zeta_2')\, ds$$

$$= |\zeta_1 - \zeta_2| + \int_0^t L\,|Z(s,\zeta_1,\mu_1^{in}) - Z(s,\zeta_2,\mu_2^{in})|\, ds + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} L\,|Z(s,\zeta_1',\mu_1^{in}) - Z(s,\zeta_2',\mu_2^{in})|\ \pi^{in}(d\zeta_1', d\zeta_2')\, ds$$