Write Thm 3.2.2, Prop 3.2.3, Thm 3.3.3 (HK1-2) on board ①

In this talk we present the ~~mea~~ main conclusion.

Thm 3.3.4 Suppose $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies (HK1-HK2).

f^{in} probability density on $\mathbb{R}^d$ s.t.

$$\int_{\mathbb{R}^d} |z| f^{in}(z)\,dz < \infty \leftarrow \text{i.e. } f^{in}\mathcal{L}^d \in \mathcal{P}_1(\mathbb{R}^d). \text{ Needed for Thm 3.2.2}$$

Then
(1) the Cauchy Problem for the mean field PDE

$$\begin{cases} \partial_t f(t,z) + \operatorname{div}_z (f(t,z)\, \mathcal{K} f(t,z)) = 0 \\ f|_{t=0} = f^{in} \end{cases}$$

has a unique weak solution $f \in C(\mathbb{R}, L^1(\mathbb{R}^d))$.

pf. $\mu^{in} = f^{in}\mathcal{L}^d \in \mathcal{P}_1(\mathbb{R}^d)$.

① By Thm 3.2.2 there is a unique soln

$$t \mapsto Z(t; \zeta^{in}, \mu^{in})$$

to the mean field characteristics eqn

$$\begin{cases} \partial_t Z(t, \zeta^{in}, \mu^{in}) = \mathcal{K}\mu(t, Z(t, \zeta^{in}, \mu^{in})) \\ \mu(t) = Z(t, \cdot, \mu^{in}) \# \mu^{in} \\ Z(0, \zeta^{in}, \mu^{in}) = \zeta^{in} \end{cases}$$

③ On the other hand, ~~by~~ to apply Exercise (6)(7) on P. 37, let $Y(t,0,\cdot) := Z(t, \cdot, \mu^{in})$ on (MF)

$b(t,z) := \mathcal{K}\mu(t,z) = \int K(z,z')\, \mu(t, dz')$

(Then $|b(t,z)| \le K(1+|z|)$ by (HK2))

$\Rightarrow$ The assumptions on the exercises are satisfied.

By Exercises (6)(7),

$$\mu(t) = Y(t,0,\cdot)\# \mu^{in}$$
$$= Z(t, \cdot, \mu^{in}) \# f^{in}\mathcal{L}^d$$

is the unique soln to

$$\begin{cases} \partial_t \mu + \mathrm{div}_z(\mu b) = 0 \\ \mu|_{t=0} = f^{in}\mathcal{L}^d \end{cases} \overset{b=K\mu}{\Longleftrightarrow} \partial_t \mu + \mathrm{div}_z(\mu K\mu) = 0$$ ②

It gives bait the mean field PDE so this is the desired conclusion.

③ $\Rightarrow$ It remains to check that the Z here is indeed the solution to the MF characteristic eqn.

By Exercise (1), $s \mapsto Y(s,t,y)$ solves

$$\begin{cases} \dot{Y}(s) = b(s, Y(s)) \\ Y(t) = y \end{cases} \quad \forall y \in \mathbb{R}^d,\ t \in [0, T]$$

Take $t := 0$, change notation $s \to t$, $y := \zeta^{in}$, we know

$$s \mapsto Y(t, 0, \zeta^{in}) = Z(t, \zeta^{in}, \mu^{in}) \text{ solves}$$

$$\begin{cases} \partial_t Y(t,0,y) = b(t, Y(t,0,y)) \\ Y(0,0,y) = y \end{cases} \Longleftrightarrow \begin{cases} \partial_t Z(t, \zeta^{in}, \mu^{in}) = K\mu(t, Z(0, \zeta^{in}, \mu^{in})) \\ Z(0, \zeta^{in}, \mu^{in}) = \zeta^{in} \end{cases}$$

This is indeed the mean field characteristic eqn. $\boxtimes$

Idea of proof:

① By Thm 3.2.2 $\exists!$ soln Z to the mean field characteristic eqn

② Z satisfies the ODE in exercise (1).

③ By Exercises (1)(6)(7), the desired soln is the unique soln to the mean field PDE

(2) (The second half of this Thm states that the empirical measure associated to the N-particle system at time t converges to

$\forall N \geq 1$, let $Z(N) = (z_{1,N}^{in}, \cdots, z_{N,N}^{in}) \in (\mathbb{R}^d)^N$ be such that

$\mu_{Z(N)} = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_{j,N}^{in}}$ (the initial empirical meas.) satisfies

$\mathrm{dist}_{MK,1}(\mu_{Z(N)}, f^{in}) \to 0$ as $N \to \infty$ → Will discuss how to achieve this in ~~next~~ second half of the talk

Let $t \mapsto T_t \mathfrak{Z}(N) = (z_{1,N}(t), \cdots, z_{N,N}(t)) \in (\mathbb{R}^d)^N$ be the soln of the N-particle ODE system with initial data $\mathfrak{Z}(N)$, i.e.

$$\begin{cases} \dot{z}_i(t) = \frac{1}{N} \sum_{j=1}^{N} K(z_i(t), z_j(t)) \ , \quad i = 1, \cdots, N \\ z_i(0) = z_i^{in} \end{cases}$$

Then $\quad \mu_{T_t \mathfrak{Z}(N)} \rightharpoonup f(t,\cdot)\mathcal{L}^d$ as $N \to \infty$ weakly.

with $\quad \mathrm{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathcal{L}^d) \le e^{2L|t|} \mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in})$

($\to 0$ by assumption)

<u>Remark</u>. The last statement indeed describes a convergence rate because $\mathrm{dist}_{MK,1}(\mu_n, \mu) \to 0 \Rightarrow \mu_n \rightharpoonup \mu$, where μ_n and μ are prob. measures, as we shall see in proof.

~~pf. $\mathrm{dist}_{MK,1}(\mu_n, \mu) = \sup_{\substack{\varphi \in Lip(\mathbb{R}^d) \\ Lip(\varphi) \le 1}} \left| \int_{\mathbb{R}^d} \varphi(z)\mu_n(dz) - \int_{\mathbb{R}^d} \varphi(z)\mu(dz) \right| \to 0$~~

~~$\Rightarrow \forall \varphi \in Lip(\mathbb{R}^d)$, $Lip(\varphi) \le 1$, $\left| \int_{\mathbb{R}^d} \varphi(z)\mu_n(dz) - \int_{\mathbb{R}^d}\varphi(z)\mu(dz) \right| \to 0$~~

~~This is part of Lem 3.3.6~~

pf of Thm. By the proof of the first part, we know that $\forall t \in \mathbb{R}$,

$$f(t,\cdot)\mathcal{L}^d = Z(t,\cdot, f^{in}\mathcal{L}^d) \# f^{in}\mathcal{L}^d \qquad (1)$$

<u>claim</u>: $\mu_{T_t\mathfrak{Z}(N)} = Z(t,\cdot,\mu_{\mathfrak{Z}(N)}) \# \mu_{\mathfrak{Z}(N)} \qquad (2) \quad \leftarrow$ discrete meas.

pf of claim:

By Prop 3.2.3, $z_{i,N}(t) = Z(t, z_{i,N}^{in}, \mu_{\mathfrak{Z}(N)}) \ \forall i = 1 \cdots N$.

Since $\mu_{T_t\mathfrak{Z}(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_{i,N}(t)}$, we know

$\mu_{T_t\mathfrak{Z}(N)}(z) = \frac{1}{N} \Leftrightarrow \exists j = 1, \cdots, N$ ~~s.t. $z = z_{j,N}(t) = Z(t, z_j$~~

s.t. $\quad z = z_{j,N}(t) = Z(t, z_{j,N}^{in}, \mu_{\mathfrak{Z}(N)})$.

On the other hand, by defn of push-forward measure, $Z(t,\cdot,\mu_{\mathfrak{Z}(N)}) \# \mu_{\mathfrak{Z}(N)}(z) = \frac{1}{N}$

$\Leftrightarrow \mu_{\mathfrak{Z}(N)}(Z(t,\cdot,\mu_{\mathfrak{Z}(N)})^{-1}(z)) = \frac{1}{N}$

$\iff Z(t,\cdot,\mu_{\mathfrak{Z}(N)})^{-1}(z) \in \mathfrak{Z}(N)$

$\iff \exists\ z_0 \in \mathfrak{Z}(N)$, s.t. $z = Z(t, z_0, \mu_{\mathfrak{Z}(N)})$.

But $\mathfrak{Z}(N) = (z^{in}_{1,N}, \cdots, z^{in}_{N,N})$ so

$\exists\ j = 1, \cdots, N$ s.t. $z = Z(t, z^{in}_{j,N}, \mu_{\mathfrak{Z}(N)})$.

Therefore the two measures take $\frac{1}{N}$ on the same set of points and 0 elsewhere, so they coincide.

This finishes the proof of the claim.

(clear the pf)

Then we directly apply Thm 3.3.3 with the measures in ① ② to see that

$$\underbrace{\mathrm{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathcal{L}^d)}_{\to 0 \text{ as } N\to\infty} \le e^{2L|t|} \underbrace{\mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in})}_{\text{By ass'n } \xrightarrow{N\to\infty} 0}$$

$\Rightarrow$

Now we show $\mu_{T_t\mathfrak{Z}(N)} \to f(t,\cdot)\mathcal{L}^d$ as $N \to \infty$.

(Write on top of new board): Recall that for (μ_n), μ probability measures, $\mu_n \to \mu$ weakly means that $\int f\, d\mu_n \to \int f\, d\mu$ for all f cts + bounded (denoted $f \in C_b$)

Portmanteau Theorem states that it suffices to show the above convergence for $f \in C_c^\infty(\mathbb{R}^d)$ because C_c^∞ functions are dense in C_b.

~~In par~~ Step 1. We show $\forall\ \varphi: \mathbb{R}^d \to \mathbb{R}$ Lipschitz,

~~Step 1. We show~~ $\left|\int_{\mathbb{R}^d} \varphi(z) \mu_{T_t\mathfrak{Z}(N)}\, dz) - \int_{\mathbb{R}^d} \varphi(z)\, f(t,z)\, dz\right| \xrightarrow{N\to\infty} 0$.

One can see this by directly applying Prop 3.3.2 which states that ($\mu_N := \mu_{T_5(N)}$, $\mu := f(t,\cdot)\mathcal{L}^d$)

$$\operatorname{dist}_{MK,1}(\mu_N, \mu) = \sup_{\substack{\varphi \in Lip(\mathbb{R}^d) \\ Lip(\varphi) \le 1}} \left| \int_{\mathbb{R}^d} \varphi(z)(\mu_N dz) - \int_{\mathbb{R}^d} \varphi(z)(\mu dz) \right|$$

$$= \sup_{\varphi \in Lip(\mathbb{R}^d)} Lip(\varphi) \cdot \left| \int_{\mathbb{R}^d} \varphi(z)(\mu_N dz) - \int_{\mathbb{R}^d} \varphi(z)(\mu dz) \right|$$

We know LHS $\to 0$ so $\forall \varphi$ Lipschitz, we have

$$\left| \int_{\mathbb{R}^d} \varphi(z)(\mu_N dz) - \int_{\mathbb{R}^d} \varphi(z)(\mu dz) \right| \to 0.$$

This justification can also be done without using this more advanced result: Let φ be Lipschitz. Then $\forall \pi \in \Pi(\mu_N, \mu)$,

$$\left| \int_{\mathbb{R}^d} \varphi(z)(\mu_N dz) - \int_{\mathbb{R}^d} \varphi(z)(\mu dz) \right|$$

by defn of coupling $\to$ $$= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi(x) - \varphi(y))\, \pi(dx\,dy) \right|$$

$$\le \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(x) - \varphi(y)|\, \pi(dx\,dy) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|\, \pi(dx\,dy) \cdot Lip(\varphi)$$

This holds for all $\pi \in \Pi(\mu_N, \mu)$, hence

$$LHS \le Lip(\varphi) \cdot \inf_{\pi \in \Pi(\mu_n, \mu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|\, \pi(dx\,dy)$$

Defn of MK dist. (Def 3.3.1) $$= Lip(\varphi) \cdot \operatorname{dist}_{MK,1}(\mu_N, \mu) \to 0$$

Step 2. We show $\forall g \in C_c(\mathbb{R}^d)$,

$$\left| \int_{\mathbb{R}^d} g(z)(\mu_N dz) - \int_{\mathbb{R}^d} g(z)(\mu dz) \right| \xrightarrow{N \to \infty} 0.$$

pf. We showed in step 1 the same relation for Lipschitz fcns. In particular, this relation holds for all $\varphi \in C_c^1(\mathbb{R}^d)$. (This is because for such φ, its derivatives are in $C_c(\mathbb{R}^d)$ so takes value in a compact subset of $\mathbb{R}$)

Let $g \in C_c(\mathbb{R}^d)$, $\varphi_n \in C_c^1(\mathbb{R}^d)$, $\varphi_n \to g$, i.e.

~~$\sup_{x \in \mathbb{R}^d} |\varphi_n(x)| \to 0$~~ $$\sup_{x \in \mathbb{R}^d} |g(x) - \varphi_n(x)| \xrightarrow{n \to \infty} 0.$$

Then $\left|\int_{\mathbb{R}^d} g(z)\,(\mu_N dz) - \int_{\mathbb{R}^d} g(z)\,(\mu dz)\right|$

$\leq \left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu_N dz)\right| + \left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu dz)\right|$

$+ \underbrace{\left|\int_{\mathbb{R}^d} \varphi_n(z)\,(\mu_N)dz - \int_{\mathbb{R}^d} \varphi_n(z)\,(\mu dz)\right|}_{\to 0 \text{ as } N\to\infty}$.

$\left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu_N dz)\right| \leq \int_{\mathbb{R}^d} |g-\varphi_n|(z)\,(\mu_N dz)$

$\leq \underbrace{\mu_N(\mathbb{R}^d)}_{=1} \underbrace{\sup_{\substack{n\in\mathbb{N}\\ x\in\mathbb{R}^d}} |g(x)-\varphi_n(x)|}_{\to 0 \text{ as } n\to\infty \text{ so is bounded}}$

Therefore DCT applies (in the $\lim_{n\to\infty}$ limit)

$\Rightarrow \lim_{n\to\infty} \left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu_N dz)\right| = 0.$

$\Rightarrow \forall \varepsilon>0, N\in\mathbb{N}\ \exists n_0$ s.t. $\left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu_N dz)\right| < \varepsilon$

$\forall\ n > n_0$.

$\Rightarrow \lim_{N\to\infty} \underbrace{\left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu_N dz)\right|}_{<\varepsilon \text{ for } n \text{ large enough}} \leq \varepsilon$.

Similar for $\lim_{N\to\infty} \left|\int_{\mathbb{R}^d} (g-\varphi_n)(z)\,(\mu dz)\right|$.

$\Rightarrow \left|\int_{\mathbb{R}^d} g(z)(\mu_N dz) - \int_{\mathbb{R}^d} g(z)\,(\mu dz)\right| \xrightarrow{N\to\infty} 0$

because ε is arbitrary.

This finishes step 2.

~~Now since $C_c(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$ dense, the Portmanteau theorem applies and hence we obtain weak convergence.~~

~~μ_N conv. weakly $\Leftrightarrow$ $\int_\Omega g\, d\mu_N$ conv. $\forall g\in D$, $D\subset C_0(\Omega)$ dense~~

~~Remark. In the proof we first showed convergence for~~ $\cancel{C_c^1}$ ~~functions. Using density of~~ $\cancel{C_c^1}$ ~~functions in~~ $\cancel{C_c}$ $\cancel{(C_c^1(\mathbb{R}^d) \subset C_c(\mathbb{R}^d))}$ ~~we extended this result to~~ $\cancel{C_c(\mathbb{R}^d)}$~~. Then the conclusion followed by density of~~ $\cancel{C_c(\mathbb{R}^d)}$ ~~in~~ $\cancel{C_b(\mathbb{R}^d)}$.
~~However~~ $\cancel{C_c^1(\mathbb{R}^d)}$ ~~this does not imply that~~ $\cancel{C_c^1(\mathbb{R}^d)}$ ~~is dense in~~ $\cancel{C_b(\mathbb{R}^d)}$ ~~because the topologies of~~ $\cancel{C_b}$ ~~and~~ $\cancel{C_c}$ ~~are defined differently.~~

Since $\int_{\mathbb{R}^d} (\mu_N dz) = \int_{\mathbb{R}^d} (\mu dz) = 1$, the above convergence holds also for all $\varphi \in C_b(\mathbb{R}^d)$, which shows weak conv. of $\mu_N \rightharpoonup \mu$ by defn.

In fact, we could have concluded with portmanteau's Thm after finishing step 1.

3.3.4 On the choice of initial data

In Thm 3.3.4 we assumed that

$$\mathrm{dist}_{MK,1}(\mu_{Z(N)}, f^{in}) \to 0 \text{ as } N \to \infty, \text{ where}$$

$$\mu_{Z(N)} = \frac{1}{N}\sum_{j=1}^{N} \delta_{z^{in}_{j,N}}$$

In this section we will show that to reach this condition, it suffices to draw an infinite sequence z_j^{in} at random according to the distribution fn $f^{in}\mathcal{L}^d$ independently.

Assume f^{in} is a prob. density on $\mathbb{R}^d$ s.t.

$$\int_{\mathbb{R}^d} |z|^2 f^{in}(z)\, dz < \infty$$

We construct a prob. space on the collection of sequences of points in $\mathbb{R}^d$ (denoted by $(\mathbb{R}^d)^{\mathbb{N}}$) as follows:

$\Omega := (\mathbb{R}^d)^{\mathbb{N}}$, σ-algebra $\mathcal{F} = \sigma\{\prod_{n\geq 1} B_n : B_n \subseteq \mathbb{R}^d$ Borel. all but finitely many B_n's are $\mathbb{R}^d$ }. (8)

$\mathbb{P} := (f^{in})^{\otimes\infty}$, i.e. on the cylinders.

$\mathbb{P}(\prod_{n\geq 1} B_n) := \prod_{n\geq 1} f^{in}(B_n)$

Then. $f^{in}(B_n) = 1$ for all but finitely many n's.

- It is easy to see $\mathbb{P}(\prod \Omega) = 1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is a prob. space.

Thm 3.3.5 For each $\vec{z}^{in} = (z_k^{in})_{k\geq 1} \in \Omega$ ($\vec{z}^{in}$ is a seq in $\mathbb{R}^d$),

let $Z_N^{in} := (z_1^{in}, \cdots, z_N^{in})$.

Then $\mathrm{dist}_{MK,1}(\mu_{Z_N^{in}}, f^{in}\mathcal{L}^d) \to 0$ as $N \to \infty$ for $\mathbb{P}$- a.s. $\vec{z}^{in} \in \Omega$.

Write lem 3.3.6 on board

pf. Consider $Y_n(\vec{z}) := \varphi(z_n)$ where $\varphi \in C_c(\mathbb{R}^d)$, or $\varphi(z) = |z|$

$\vec{z} = (z_1, \cdots, z_n, \cdots) \in \Omega$.

Then $Y_n : \Omega \to \mathbb{R}$ measurable so Y_n is a random variable.

We will apply SLLN on $(Y_n)_{n\geq 1}$ so we check iid + L^1:

(1) Y_n's are identically distributed

pf. Let $a \in \mathbb{R}$ be arbitrary. Then

$$\mathbb{P}(Y_n \geq a) = \mathbb{P}(\{\vec{z} \in (\mathbb{R}^d)^{\mathbb{N}} : \varphi(z_n) \geq a\})$$

$$= \mathbb{P}(\prod_{j\geq 1}\{B_j : B_j = \mathbb{R}^d \text{ for } j \neq n;\ B_n = \{z_n : \varphi(z_n) \geq a\}\})$$

$$= \mathbb{P}(\prod_{j\geq 1}\{B_j : B_n = \varphi^{-1}([a,\infty)),\ B_j = \mathbb{R}^d\ \forall j \neq n\})$$

By defn of $\mathbb{P}$ $\to$ $$= \prod_{\substack{j\geq 1\\ j\neq n}} f^{in}(\mathbb{R}^d) \cdot f^{in}(\varphi^{-1}([a,\infty)))$$

$$= \int \mathbb{1}_{\{\varphi^{-1}([a,\infty))\}}(z)\, f^{in}(z)\, dz$$

This does not depend on n.

(9)

(2) Y_n's are indep.

pf. ~~By defn of independence of a seq. of r.v.'s, we need to show~~

It suffices to show

$\mathbb{E}_{\mathbb{P}}[g_1(Y_1)\cdots g_N(Y_N)] = \prod_{k=1}^{N} \mathbb{E}_{\mathbb{P}}[g_k(Y_k)]$ where

$g_1, \cdots, g_N \in C_b(\mathbb{R})$.

$$\mathbb{E}[g_1(Y_1)\cdots g_N(Y_N)] = \int_\Omega g_1(Y_1)\cdots g_N(Y_N)\, d\mathbb{P}$$

By def of $\mathbb{P}$

$$= \int_\Omega g_1(\varphi(z_1))\cdots g_N(\varphi(z_N)) \prod_{n\geq 1} f^{in}(z_n)\, dz_n$$

$$= \int_{(\mathbb{R}^d)^{\{N+1,N+2,\cdots\}}} \int_{(\mathbb{R}^d)^N} g_1(\varphi(z_1))\cdots g_N(\varphi(z_N)) \prod_{n=1}^{N} f^{in}(z_n)\, dz_n \prod_{n\geq N+1} f^{in}(z_n)\, dz_n$$

Fubini (g bounded)

$$\to = \int_{(\mathbb{R}^d)^{\{N+1,N+2,\cdots\}}} \left(\prod_{n=1}^{N} \int_{\mathbb{R}^d} g_n(\varphi(z_n))\, f^{in}(z_n)\, dz_n \right) \prod_{n\geq N+1} f^{in}(z_n)\, dz_n$$

Fubini

$$\to = \prod_{n=1}^{N} \int_\Omega g_n(Y_n(\vec{z}))\, d\mathbb{P}(\vec{z}) \cdot \prod_{n\geq N+1} \underbrace{f^{in}(\mathbb{R}^d)}_{=1}$$

$$= \prod_{n=1}^{N} \mathbb{E}[g_n(Y_n)].$$

$\Rightarrow$ Y_n's are indep.

(3) $\mathbb{E}[|Y_k|^2] < \infty \quad \forall k \in \mathbb{N}$

$$\mathbb{E}[|Y_k|^2] = \int_\Omega |Y_k(\vec{z})|^2\, d\mathbb{P}(\vec{z})$$

$$= \int_{(\mathbb{R}^d)^{\mathbb{N}}} |\varphi(z_k)|^2 \prod_{n\geq 1} f^{in}(z_n)\, dz_n$$

Use Tonelli ~~Fubini~~ as above

$$\to = \int_{\mathbb{R}^d} |\varphi(z)|^2 f^{in}(z)\, dz < \infty$$

~~($\varphi \in C_c$)~~ (If $\varphi \in C_c$, $\leq \left(\sup_{z\in\mathbb{R}} |\varphi(z)|\right)^2 < \infty$; If $\varphi(z) = |z|$, use ass'n)

Therefore SLLN applies, that is, $\sum_{k=1}^{N} Y_k \to \mathbb{E}[Y_1]$ a.s.

$\mathbb{E}[Y_1] = \int_\Omega Y_1(\vec{z})\, d\mathbb{P}(\vec{z}) = \int_{\mathbb{R}^d} \varphi(z_1) f^{in}_{(z_1)} dz_1 = \int_{\mathbb{R}^d} \varphi(z) f^{in}_{(z)} dz$ 10

$\frac{1}{N}\sum_{k=1}^N Y_k = \frac{1}{N}\sum_{k=1}^N \varphi(z_k) = \frac{1}{N}\sum_{k=1}^N \delta_{z_k}(\varphi) = \frac{1}{N}\sum_{k=1}^N \langle \delta_{z_k}, \varphi\rangle$

Therefore $\frac{1}{N}\sum_{k=1}^N \langle \delta_{z_k}, \varphi\rangle \to \underbrace{\int_{\mathbb{R}^d} \varphi(z) f^{in}(z)\, dz}_{= \langle f^{in}\mathcal{L}^d, \varphi\rangle}$ for a.e. $\vec{z}\in\Omega$.

This holds for all $\varphi \in C_c(\mathbb{R}^d)$ and $\varphi(\cdot) = |\cdot|$ (*)

Since $C_c(\mathbb{R}^d)$ is separable (by Stone-Weierstrass), we can assume the $\mathbb{P}$-negligible set is same for all φ.

$\Rightarrow \frac{1}{N}\sum_{k=1}^N \langle \delta_{z_k}, \varphi\rangle \to \int_{\mathbb{R}^d}\varphi(z) f^{in}(z)\, dz \quad \forall \varphi\in C_c(\mathbb{R}^d)$ or $\varphi(\cdot)=|\cdot|$ (*)

for a.e. $\vec{z}\in\Omega$. So by defn

$$\frac{1}{N}\sum_{k=1}^N \delta_{z_k} \to f^{in}\mathcal{L}^d$$

weakly (conv. in measure)

Can be simplified See P13 (+)

Furthermore, $\sum_{k=1}^N \delta_{z_k} \in \mathcal{P}_1(\mathbb{R}^d)$, $f^{in}\mathcal{L}^d \in \mathcal{P}_1(\mathbb{R}^d)$, and

$\sup_N \int_{\mathbb{R}^d} |z|\, \mathbb{1}_{|z|\geq R} (\frac{1}{N}\sum_{k=1}^N \delta_{z_k}\, dz)$

$= \sup_N \frac{1}{N}\sum_{k=1}^N |z_k|\, \mathbb{1}_{|z_k|\geq R}$

By (*) for $\varphi(\cdot) = |\cdot|$ we have

$\langle\frac{1}{N}\sum_{k=1}^N \delta_{z_k}, \varphi\rangle = \frac{1}{N}\sum_{k=1}^N \varphi(z_k) = \frac{1}{N}\sum_{k=1}^N |z_k| \to \int_{\mathbb{R}^d} f^{in}(z)\, dz < \infty$

$\Rightarrow \exists N_0$ s.t. $\langle \frac{1}{N}\sum_{k=1}^N \delta_{z_k}, \varphi\rangle < 1 + \int_{\mathbb{R}^d} f^{in}(z)\, dz$. indep. of R

$\Rightarrow \sup_{N\geq N_0} \int_{\mathbb{R}^d} |z|\mathbb{1}_{|z|>R}\, \mu_n\, dz = \sup_{N\geq N_0} \frac{1}{N}\sum_{k=1}^N |z_k|\mathbb{1}_{|z_k|\geq R} \leq 1 + \int_{\mathbb{R}^d} f^{in}(z)dz < \infty$

DCT $\Rightarrow \lim_{R\to\infty}\sup_{N\geq N_0}\int_{\mathbb{R}^d} \cdots \mu_n\, dz \to \sup_{N>N_0}\int_{\mathbb{R}^d} \lim_{R\to\infty} |z|\underbrace{\mathbb{1}_{|z|>R}}_{\to 0}\, dz = 0.$

~~That~~ $\sup_{N\leq N_0}\int_{\mathbb{R}^d} \cdots \mu_n\, dz = \sup_{N\leq N_0}\frac{1}{N}\sum_{k=1}^N |z_k|\underbrace{\mathbb{1}_{|z_k|\geq R}}_{} \to 0$ as $R\to\infty$

if $R > \max\{|z_1|, \cdots |z_{N_0}|\}$

Therefore $\sup_N \int_{\mathbb{R}^d} \cdots \mu_n\, dz \to 0$ as $R\to\infty$

$\Rightarrow$ lem 3.3.6 (2) is satisfied

$\Rightarrow$ By lem 3.3.6 (2) $\Rightarrow$ (1) we conclude.

pf of (*). We follow the proof given in notes P49-50.

Let $R>0$. $E_R = \{ \varphi \in C([-R,R]^d, \mathbb{R}^d) : \varphi|_{\partial[-R,R]^d} = 0 \}$

$\|\varphi\| := \sup_{x\in[-R,R]^d} |\varphi(x)|$

claim 1. $(E_R, \|\cdot\|)$ is a separable Banach space

pf. Separability is a consequence of Stone-Weierstrass

Completeness: Let (φ_n) be a Cauchy seq.

i.e. $\sup_{x\in[-R,R]^d} |\varphi_n(x) - \varphi_m(x)| \to 0$ as $m,n \to \infty$.

Then $(\varphi_n(x))_n$ is a Cauchy seq in $\mathbb{R}$ so converges.

Define $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$.

(1) $\varphi(x)$ is cts because ~~$\varphi(x-y)=$~~

$|\varphi(x)-\varphi(y)| \le |\varphi(x)-\varphi_n(x)| + |\varphi_n(x)-\varphi_n(y)| + |\varphi(y)-\varphi_n(y)|$

$\exists N>0$ s.t. $|\varphi(x)-\varphi_n(x)| < \varepsilon$, $|\varphi(y)-\varphi_n(y)| < \varepsilon$ $\forall n \ge N$.

$\Rightarrow |\varphi(x)-\varphi_N(x)| < \varepsilon$, $|\varphi(y)-\varphi_N(y)| < \varepsilon$

For x,y suff. close, $|\varphi_N(x)-\varphi_N(y)| < \varepsilon$.

$\Rightarrow |\varphi(x)-\varphi(y)| < 3\varepsilon$ for x,y suff. close.

$\Rightarrow \varphi$ is cts.

$\Rightarrow \varphi \in E_R$.

(2) $\|\varphi_n - \varphi\| \xrightarrow{n\to\infty} 0$. i.e. φ is indeed the limit of φ_n

~~$\lim_{n\to\infty} \|\varphi_n - \varphi\| = \lim_{n\to\infty} \|\varphi_n$~~

Since φ_n and φ are cts fns on $[-R,R]^d$ compact, and $\varphi_n \to \varphi$ pointwise (by defn of φ),

$\varphi_n \to \varphi$ uniformly. i.e. $\sup_{x\in[-R,R]^d} |\varphi_n(x) - \varphi(x)| \to 0$.

$\Rightarrow \|\varphi_n - \varphi\| \xrightarrow{n\to\infty} 0$, as desired.

Now define $N_{\varphi,R} := \{ \vec{z} \in \Omega : \langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \varphi \rangle \not\to \int_{\mathbb{R}^d} \varphi(z) f^{in}(z)\, dz \} \subseteq \Omega$.

~~Defn~~ Let $(\varphi_n)_{n\ge1}$ be a countable dense set of E_R.

Define $N_R := \bigcup_{n\ge1} N_{\varphi_n,R}$

claim 2. $\langle \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}, \varphi \rangle \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} \varphi(z) f^{in}(z)\,dz \quad \forall \varphi \in E_R,\ \vec{z} \notin N_R.$

pf. By density of (φ_n), $\exists\, m = m(\varphi,\varepsilon) \in \mathbb{N}$ s.t. $\|\varphi - \varphi_m\| < \varepsilon$.

$\vec{z} \notin N_R \Leftrightarrow \vec{z} \in \Omega$ s.t. $\langle \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}, \varphi_n \rangle \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} \varphi_n(z) f^{in}(z)\,dz \ \forall n \in \mathbb{N}.$

Let $\varepsilon > 0$.

$\Rightarrow |\text{LHS} - \text{RHS}| \le \underbrace{\left|\langle \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}, \varphi - \varphi_m \rangle\right|}_{\le \sup|\varphi - \varphi_m| = \varepsilon} + \underbrace{\left|\int_{\mathbb{R}^d}(\varphi_m(z) - \varphi(z)) f^{in}(z)\,dz\right|}_{\le \sup|\varphi_m - \varphi| = \varepsilon}$

because $\frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}$ and $f^{in}(z)\,\mathcal{L}^d$ are prob. meas.

$+ \underbrace{\left|\langle \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}, \varphi_m \rangle - \int_{\mathbb{R}^d}\varphi_m(z) f^{in}(z)\,dz\right|}_{< \varepsilon \text{ for } N \text{ large enough by the choice of } \vec{z}}$

$< 3\varepsilon$ for N large enough.

ε is arbitrary $\Rightarrow$ claim 2 is verified.

In the context of Thm 3.3.5, let R be arbitrary, we had in $(*)$ (Page 10) that $\mathbb{P}(N_{\varphi_n,R}) = 0 \quad \forall n \in \mathbb{N}$.

$\Rightarrow \mathbb{P}(N_R) = \mathbb{P}(\bigcup_{n\ge 1} N_{\varphi_n,R}) = 0$.

By claim 2, $\langle \frac{1}{N}\sum_{k=1}^{N}\delta_{z_k}, \varphi \rangle \xrightarrow{N\to\infty} \int_{\mathbb{R}^d}\varphi(z) f^{in}(z)\,dz \ \forall \varphi \in E_R, \vec{z} \notin N_R$.

Since this is true for R arbitrary, the statement holds for all

$\varphi \in \bigcup_{R=1}^{\infty} E_R \supseteq C_c(\mathbb{R}^d)$ provided $\vec{z} \notin \underbrace{\bigcup_{R=1}^{\infty} N_R}$

$\mathbb{P}(\ ) = 0$

This is the $\mathbb{P}$-negligible set we take

(†) An easier way to show

$$\sup_N \int_{\mathbb{R}^d} |z| \mathbb{1}_{|z|\geq R} \mu_N dz \to 0 \quad \text{as} \quad R \to \infty \quad (*):$$

Define $\tilde{\varphi}(z) := |z| \mathbb{1}_{|z|\geq R}$.

Then if we again let $Y_n(\vec{z}) := \tilde{\varphi}(z_n)$, SLLN still applies

and $\underbrace{\langle \frac{1}{N}\sum_{k=1}^{N} \delta_{z_k}, \tilde{\varphi} \rangle}_{} \Rightarrow \int_{\mathbb{R}^d} \tilde{\varphi}(z) \, f^{in}(z)\, dz$

$$\frac{1}{N}\sum_{k=1}^{N}\int |z| \mathbb{1}_{|z|\geq R} d(\delta_{z_k}) \qquad \int_{\mathbb{R}^d} \underbrace{|z| \mathbb{1}_{|z|\geq R} f^{in}(z) dz}_{\to 0}$$

$$= \frac{1}{N}\sum_{k=1}^{N} |z_k| \mathbb{1}_{|z_k|\geq R} = \text{LHS of } (*) . \quad \xrightarrow{R\to\infty} 0 \text{ By DCT}$$

$\Rightarrow$ As $N \to \infty$, LHS of $(*)$ $\to 0$ as $R \to \infty$.

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists N_0$ s.t. for R large enough, LHS $< \varepsilon \; \forall N \geq N_0$

For $N < N_0$ LHS $\xrightarrow{R\to\infty} 0$ directly by DCT.

$$\Rightarrow \sup_N \int_{\mathbb{R}^d} |z| \mathbb{1}_{|z|\geq R} \mu_N dz \to 0 \text{ as } R \to \infty .$$