Only the exercises with an asterisk (*) will be corrected.

3.1. MC questions

(a) We consider the ODE

$$u''(x) - 4u(x) = -1\,, \quad x \in (0,1)$$

with $u(0) = 0$ and $u'(1) + u(1) = 0$. What is the maximal value of $u(x)$?

- ☐ 0
- ☐ $\frac{\sqrt{3e^6+2e^4-e^2}}{e^4+1}$
- ☐ $e^6 + 2e^4 - e^2$
- ☐ $-\frac{1}{2}\frac{\sqrt{3e^6+2e^4-e^2}}{3e^4+1} + \frac{1}{4}$
- ☐ $-\frac{1}{4}\frac{\sqrt{3e^6+2e^4-e^2}}{e^4+1} + \frac{1}{2}$.

(b) Consider the following differential equation

$$y^{(4)} + 2y^{(2)} = 0. \qquad (\spadesuit)$$

Which of the following statements are true?

- ☐ The space of solutions of ($\spadesuit$) satisfying $y(0) = 0$ is a 3-dimensional vector space.
- ☐ The space of solutions of ($\spadesuit$) satisfying $y(0) = 1$ is a 3-dimensional vector space.
- ☐ The space of solutions of ($\spadesuit$) satisfying $\lim_{t\to\infty} y(t) = 0$ is a 3-dimensional vector space.
- ☐ The space of solutions of ($\spadesuit$) satisfying $y'(0) = 0$ is a 2-dimensional vector space.

3.2. A glance at systems.

Let

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad t \in I \subseteq \mathbb{R} \qquad (*)$$

be a linear, homogeneous ODE of order $n \geq 2$ with constant coefficients.

(a) Show that, by setting $z_1 = y$, $z_2 = y'$, …, $z_n = y^{(n-1)}$, equation $(*)$ can be seen as a first-order *system* of ODEs:

$$\boldsymbol{z}' = A\boldsymbol{z}, \quad t \in I \tag{$\star$}$$

where $A \in M_{n\times n}(\mathbb{R})$ is a $n \times n$ matrix. Write down explicitly the expression for A.

(b) Prove that the characteristic polynomial of the ODE $(*)$ is the characteristic polynomial of the matrix A.

(c) Show that

$$\boldsymbol{\zeta}(t) = \mathrm{e}^{\lambda t}\boldsymbol{u}$$

is a solution of the homogeneous problem $(\star)$ if and only if λ is an eigenvalue of A and $\boldsymbol{u}$ is a corresponding eigenvector.

(d) *Fact:* Similarly as for ODE of order n, one can prove that for a system like $(\star)$, the set of solutions is vector space of dimension n.

Assuming that all the eigenvalues of A are distinct, use (c) and the fact above to find an explicit expression for the general solution of homogeneous system.

*3.3. Solving ODEs

(a) Determine the general solution of the ODE

$$y''' + 5y'' - y' - 5y = 0.$$

(b) Determine the general solution of the ODE

$$y'' + y' = 0$$

with initial condition $y(1) = y'(1) = 2$.

(c) The ODE

$$f'' + 2qf' + (q + q^2)f = 0$$

contains a real parameter q. For which values of q do **all** the solutions remain bounded for $x \to \infty$?
Hint: Distinguish carefully the different cases concerning the zeros of the characteristic polynomial!

(d) Determine the general (real) solution of the ODE

$$y^{(4)} + 2y^{(2)} + y = 0,$$

which satisfies the initial conditions

$$y(0) = y'(0) = y''(0) = 0, \qquad y^{(3)}(0) = 1\,.$$

***3.4. ODE with given solutions.**

(a) Find a linear ODE with constant coefficients such that e^{-x}, e^{x}, $e^{-\pi x}$ and $e^{\pi x}$ are solutions of the equation.

(b) Find a linear ODE with constant coefficients such that e^{-10x} and $e^{3x}\cos(3x)$ are solutions of the equation.