Only the exercises with an asterisk (*) will be corrected.

6.1. MC questions.

(a) Which of the following statements are true?

☐ Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and p is a integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be function. If $g \circ f$ is continuous, then g is continuous or f is continuous.

☐ Let $X \subset \mathbb{R}^n$ be closed and p is a integer. If $f : X \to \mathbb{R}^p$ is a continuous function, then $f(X)$ is closed.

(b) Which of the following sets are compact?

☐ $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 2022\}$;

☐ $B = \{(a, b, c) \in \mathbb{R}^3 \,|\, a, b, c \text{ are integers and } a^2 + b^2 + c^2 < 2022\}$;

☐ $C = \{(x, f(x)) \in \mathbb{R}^2 \,|\, x \in (0, 1], f(x) = \sin(\frac{1}{x})\}$;

☐ $D = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \,|\, \theta \text{ is a rational number}\}$;

☐ $E = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq 2\}$.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. For f to be continuous at $(0, 0)$, which of the following needs to hold?

Hint: it might be helpful to do Exercise 6.3 first.

☐ it is sufficient that, along *some* direction $v \neq 0$, the directional derivative $D_v f(0, 0)$ exists.

☐ it is sufficient that *both* the partial derivatives $\partial_x f(0, 0)$ and $\partial_y f(0, 0)$ exist.

☐ it is sufficient that the directional derivatives $D_v f(0, 0)$ along *every* direction $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ exists.

***6.2. Jacobi matrix** Compute the Jacobi matrix of the following functions:

(a)

$$f : \mathbb{R}^3 \to \mathbb{R}^2\,, \quad (u, v, w) \mapsto \begin{pmatrix} uv \\ w \end{pmatrix}.$$

(b)

$$g : \mathbb{R}^2 \to \mathbb{R}^3\,, \quad (x,y) \mapsto \begin{pmatrix} x^2 + e^y \\ x + y \\ y \end{pmatrix}$$

(c)

$$h : \mathbb{R}^3 \to \mathbb{R}^3; \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\,\cos(\theta)\cos(\phi) \\ r\,\cos(\theta)\sin(\phi) \\ r\,\sin(\theta) \end{pmatrix}\,.$$

***6.3. Partial derivatives vs. differentiability.**

Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a function and $v \in \mathbb{R}^n$ a vector. When it exists, the limit

$$D_v g(x) = \lim_{h\to 0} \frac{g(x+hv) - g(x)}{h}$$

is called the *directional derivative* of g along v at the point $x \in \mathbb{R}^n$. In particular, along the coordinate directions $e_i = (0,\dots,0,1,0,\dots,0)$ we have $D_{e_i} g = \frac{\partial g}{\partial x_i} = \partial_{x_i} g$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ to be the following function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that, for any point $a \in \mathbb{R}^2$ and any direction $u \in \mathbb{R}^2$, f admits a directional derivative $D_u f(a)$.

Hint: To prove $D_u f(a)$ exists, one needs to show that $t \mapsto f(a+tu)$ is differentiable at the point $t = 0$. Recall the definition of differentiability in one variable and use the value $f(0,0)$.

(b) Show that f is not differentiable at the point $(0,0)$.

Hint: Recall that, when f is differentiable at the point $a \in \mathbb{R}^2$, f is also continuous at a.

6.4. Tangent plane. Given the function

$$\begin{aligned} f\colon \Omega &\to \mathbb{R} \\ (x,y) &\mapsto \sqrt{1 - x^2 - y^2}, \end{aligned}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, compute the tangent plane of the graph of f at the points $(0,0)$, $\left(\frac{\sqrt{2}}{2}, 0\right)$, and $\left(0, \frac{1}{2}\right)$.