Only the exercises with an asterisk (*) will be corrected.

7.1. MC questions.

(a) Determine whether the following statement is true or false.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then, without doing any computation, we always know what is its differential $dL(x)$ at every point $x \in \mathbb{R}^m$.

☐ True ☐ False

(b) Determine whether the following statements is true or false.

Let $M_{n\times n}(\mathbb{R})$ be the space of $n \times n$ matrices which we identify with the Euclidean space $\mathbb{R}^{n^2}$. The function "determinant" $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$, is

		True	False
(a)	is continuous on $M_{n\times n}(\mathbb{R})$	☐	☐
(b)	continuous only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐
(c)	differentiable on $M_{n\times n}(\mathbb{R})$	☐	☐
(d)	differentiable only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐

*7.2. Calculating the differential.

Calculate $df(x_0)[v]$ for

(a) $f(x) = x^2, x_0 = 3, v = 5.$

(b) $f(x) = y\cos^2(x), x_0 = (\pi/4, 1), v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) $f(x) = e^x \ln(y) + z, x_0 = (0, 1, 0), v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

7.3. Gradient and Level Sets.

A *curve in the plane* is a subset $\Gamma \subset \mathbb{R}^2$ so that there is a differentiable function $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^2$ so that $\gamma((a, b)) = \Gamma$ and $\gamma'(t) \neq 0$ for every $t \in (a, b)$. Any such γ is called *parametrization* of Γ.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function so that its *level set* at c

$$\Gamma = f^{-1}(\{c\}) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$$

is a curve in the plane, and let $\gamma : I = (a, b) \to \Gamma$ be a parametrization. Prove that:

(a) The gradient to f is orthogonal to Γ, namely

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \text{for every } t \in I.$$

(b) The directional derivative of f along Γ vanishes, namely

$$D_{\gamma'(t)} f(\gamma(t)) = 0 \quad \text{for every } t \in I.$$

(c) At a point $(x, y) \in \Gamma$, the direction where f grows the most is orthogonal to Γ (this means: among all the vectors v with $|v| = 1$, $D_v f(x, y)$ assumes the maximum value when v is orthogonal to Γ).

***7.4. Calculations in polar coordinates.**

The polar coordinates on $\mathbb{R}^2$ are defined via the map

$$P : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2, \qquad P(r, \phi) := \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}.$$

(a) Show that P is continuously differentiable and calculate the derivative dP.

(b) Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ be a differentiable function and define $g(r, \phi) := f(P(r, \phi))$. Show that

$$\partial_x f(P(r, \phi)) = \partial_r g(r, \phi)\cos(\phi) - \frac{1}{r}\sin(\phi)\partial_\phi g(r, \phi)$$
$$\partial_y f(P(r, \phi)) = \partial_r g(r, \phi)\sin(\phi) + \frac{1}{r}\cos(\phi)\partial_\phi g(r, \phi).$$

holds and conclude that

$$\nabla f(P(r, \phi)) = \partial_r g(r, \phi) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} + \frac{1}{r}\partial_\phi g(r, \phi) \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix}.$$

Hint: *Use the chain rule.*

(c) Let

$$f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}, \qquad f(x, y) := \log(\sqrt{x^2 + y^2}).$$

Use the chain rule and part (b) to verify that

$$\nabla f(x, y) = \frac{1}{x^2 + y^2} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hint: *the function f is in polar coordinates given by $g(r, \phi) = \log(r)$.*