Only the exercises with an asterisk (*) will be corrected.

8.1. MC questions.

(a) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^k map with $k \geq 2$. Which of the following statements are true?

- ☐ The gradient $\nabla f(x)$ is an $n \times n$ matrix.
- ☐ The Hessian matrix $\mathrm{Hess}_f(x)$ is a square matrix.
- ☐ $\mathrm{Hess}_f(x)$ is symmetric.
- ☐ $\mathrm{Hess}_f(x)$ is invertible.

(b) Consider the function $f(x, y) := (x^2 + y^2)e^{xy}$. Which statements are correct?

- ☐ The Hessian matrix of f at $(0, 0)$ is positive definite.
- ☐ The Hessian matrix of f at $(0, 0)$ is negative definite.
- ☐ The Hessian matrix of f at $(0, 0)$ has positive and negative eigenvalues.

8.2. Hessian matrix.

Calculate the Hessian matrix of the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point (x_0, y_0):

(a) $f(x, y) = x^2 + xy + y^2, \quad (x_0, y_0) = (1, 1)$,

(b) $f(x, y) = \cos(x) + \sin(y)x, \quad (x_0, y_0) = (\pi/2, \pi/2)$,

(c) $f(x, y) = y^3 \cdot \left(\sin\left(e^{y^5 \cdot \sin(y^2)}\right) \cdot \cos(\sinh(y))\right), \quad (x_0, y_0) = (0, 0)$.

*8.3. Taylor polynomials.

Consider the following function:

$$f(x, y) := e^{x^2+y^2} + \log(1 + x^2) + \arctan(xy).$$

(a) Determine the Taylor polynomial of f at $(0, 0)$ up to and including third order.

(b) Let

$$g\colon \mathbb{R} \to \mathbb{R}, \qquad g(x) := f(x, x).$$

Determine the Taylor polynomial of g at $(0, 0)$ up to and including third order.

What is the relationship between the Taylor polynomials of f and of g?

***8.4. Taylor approximation.**

Approximate the function

$$f(x, y) = e^x \sin y$$

at the point $(0, \pi/2)$ by a Taylor polynomial of first order and of second order. Use these Taylor polynomials to give an approximation of $f\left(0,\ \frac{\pi}{2} + \frac{1}{4}\right)$.