Only the exercises with an asterisk (*) will be corrected.

10.1. MC questions.

(a) Let $X \subseteq \mathbb{R}^n$ be an open set, $f\colon X \to \mathbb{R}^n$ a C^1 function, $\gamma\colon [a,b] \to X$ a parametrised path. Which of the following statements are true?

- ☐ If $g\colon X \to \mathbb{R}$ is a potential of f, then for every constant $C \in \mathbb{R}$, $h := g + C$ is also a potential of f.
- ☐ The vector field f is conservative if and only only if f has a potential g.
- ☐ If X is star-shaped, then f is conservative.
- ☐ If for every $i, j \in \{1, \dots, n\}$ the equation $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ holds, then f is conservative.
- ☐ Let $A_1, \dots, A_m \subseteq X$ be open with $\bigcup_{k=1}^m A_k = X$. If $f|_{A_k}$ is conservative for all $k = 1, \dots, m$, then f is conservative.

(b) The work W done by a force F on a body moving along a path γ is given by the line integral

$$W = \int_\gamma F(s) \cdot d\gamma.$$

Consider a spring that exerts a horizontal force $F\colon \mathbb{R}^2 \to \mathbb{R}^2, F(x,y) = (-kx, 0)$. Is the following statement true or false?

The work W done by this spring on a body moving along the path $\gamma(t) = (x(t), y(t))$ is given by

$$W = -\frac{kx^2}{2}.$$

- ☐ True.
- ☐ False.

*10.2. Line integrals.

Compute the following line integrals.

(a) $v(x,y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$, from $(-1,1)$ to $(1,1)$ along the curve $y = x^2$.

(b) $v(x,y) = \begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix}$, from $(0,0)$ to $(2,0)$ along the curve $y = 1 - |1-x|$.

(c) $v(x,y,z) = \begin{pmatrix} x \\ y \\ xz-y \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} t^2 \\ 2t \\ 4t^3 \end{pmatrix}$, $t \in [0,1]$.

(d) $v(x,y) = \begin{pmatrix} 2a-y \\ x \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} a(t-\sin(t)) \\ a(1-\cos(t)) \end{pmatrix}$, $t \in [0, 2\pi]$, with a constant $a \in \mathbb{R}$.

10.3. Magnetic field.

The following vector field describes, according to the *Biot–Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the z-axis:

$$B(x,y,z) = \frac{\mu_0 I}{2\pi} \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x,y) \neq 0,$$

where μ_0 and I are, respectively, the magnetic constant and I the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i} B_j = \frac{\partial}{\partial x_j} B_i \quad \forall\, i,j \in \{1,2,3\},$$

where we denoted $(x_1, x_2, x_3) = (x,y,z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Is B conservative? Does B admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?

*10.4. Vector field

(1) Check that the vector-field $\mathbb{R}^2$

$$f(x,y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$$

is conservative.

(2) Compute a potential of f.

(3) Compute

$$\int_\gamma f \cdot d\vec{s},$$

where γ is the parametrised curve

$$\begin{cases} \gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \mathbb{R}^2 \\ \theta \mapsto \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(\theta), \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\theta)\right) \end{cases}$$

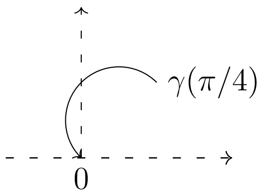

Figure 1: Curve γ.

shown in Figure 1.