Only the exercises with an asterisk (*) will be corrected.

11.1. MC questions.

(a) Which of the following statements are true?

Recall that, for a C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, its gradient $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ can be thought as a vector field. The equation

$$\operatorname{curl}(\nabla f) = g$$

- ☐ has a solution f for *every* given g
- ☐ has a solution f *only* for $g = 0$
- ☐ has a solution f *also* for some nonzero g's.

(b) Which of the following statements are true?

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, regular region. Generally speaking, the integral $\int_\Omega d\mu$ represents:

- ☐ the area of Ω
- ☐ the length of the curve bounding Ω
- ☐ the volume of a cylinder with base Ω and height 1
- ☐ the surface area of some cylinder with base Ω and height 1.

*11.2. Integrability.

Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) = \left(\frac{2k-1}{2^n}, \frac{2l-1}{2^n}\right) \text{ for } k,\, l \in \mathbb{N} \text{ and } n \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases}$$

a) Using upper and lower sums, show that f is not integrable.

b) Show that for all $x \in [0,1]$ the function $y \mapsto f(x,y)$ is integrable with

$$\int_0^1 f(x,y)\, dy = 0$$

and therefore $\int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dx = 0$.

***11.3. Double integrals.**

Compute the following integrals:

(a) $\displaystyle\int_0^1 \int_0^x e^{x+y}\, dy\, dx$

(b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 x \cos y\, dx\, dy$

(c) $\displaystyle\int_1^3 \int_2^{5-x} \frac{1}{(x+y)^3}\, dy\, dx.$

Now describe and draw the domains of integration, and compute the integrals exchanging order of integration. Is the result the same?

11.4. Fubini's theorem for explicit functions.

(a) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4 y - y^5 x + y^3\right) dxdy$$

(b) Let $D^2 = \mathbb{R}^2 \cap \{(x,y) : x^2 + y^2 \leq 1\}$ be the unit disk in the plane. Compute

$$\int_{D^2} x^2 y^2 dxdy$$

by following the steps below.

(I) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x,y) dxdy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) dy \right) dx.$$

(II) Show that

$$\int_{D^2} x^2 y^2 dxdy = \frac{4}{3} \int_0^{\frac{\pi}{2}} \cos^4(\theta) \sin^2(\theta) d\theta$$

by using the formula from the previous step and by making a change of variables $x = \sin(\theta)$.

(III) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta) \sin^2(\theta) d\theta = \frac{\pi}{32}$$

and calculate $\int_{D^2} x^2 y^2 dxdy$.