Only the exercises with an asterisk (*) will be corrected.

12.1. MC questions.

(a) Choose the correct statement. Motivate your answer.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous functions and let $B_r(0) \subset \mathbb{R}^n$ be the ball of radius $r > 0$ centred ad the origin. The integral

$$\int_{B_r(0)} f(x)\, dx$$

can also be written as

☐ $r^n \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$

☐ $\frac{1}{r^n} \int_{B_1(0)} f(rx)\, dx$

☐ $r^n \int_{B_1(0)} f(rx)\, dx$

☐ $\frac{1}{r^n} \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$

where we denoted $rx = (rx_1, \dots, rx_n)$ and similarly for $\frac{1}{r}x$.

(b) Which of the following vector fields $f\colon X \to \mathbb{R}^n$ admit a potential?

☐ $X = \mathbb{R}^2, \qquad f(x,y) = \begin{pmatrix} x \\ xy \end{pmatrix},$

☐ $X = \mathbb{R}^2 \setminus \{0\}, \qquad f(x,y) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

☐ $X = \mathbb{R}^2, \qquad f(x,y) = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$

☐ $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}, \qquad f(x,y,z) = \begin{pmatrix} e^z \sin(z) x \\ 0 \\ \frac{1}{2}x^2 e^z(\cos(z) + \sin(z)) \end{pmatrix}$

***12.2. Volume of the region enclosed by graphs of functions.**

Let

$$K_1 = \left\{(x, y, z) : 1 \leq z < \infty,\ \sqrt{x^2 + y^2} \leq \sqrt{-1 + z}\right\},$$

and

$$K_2 = \left\{(x, y, z) : -\infty < z \leq 5,\ \sqrt{x^2 + y^2} \leq \sqrt{5 - z}\right\}.$$

Calculate the volume of $K_1 \cap K_2$.

12.3. Line integral vs double integral of curl.

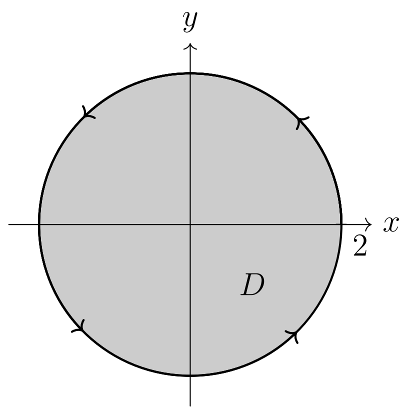
The *curl* of a vector field in $\mathbb{R}^2$ is, by definition, the function

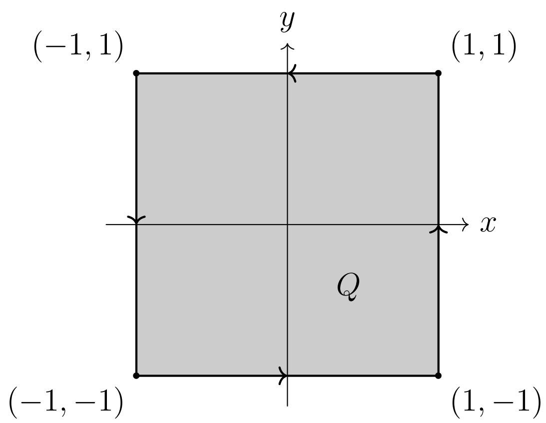
$$\operatorname{curl}(v) = \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1.$$

Consider the vector field $v(x, y) = (y^2, x)$.

(a) Compute the line integral of v along the circle of radius 2 centered at the origin and along the square of vertices $(\pm 1, \pm 1)$, both oriented counter-clockwise (see the picture).

(b) Now compute the double integral of $\operatorname{curl}(v)$ over the disk D and the square Q enclosed by the curves in (b). What do you notice?





***12.4. Volume of a 3-dimensional ball.**

Let $r > 0$ and $B_3(0, r) = \mathbb{R}^3 \cap \{(x, y, z) : x^2 + y^2 + z^2 \leq r^2\}$ be the open ball of radius $r > 0$. By using a change of coordinates, $f\colon [0, r) \times [0, 2\pi) \times [0, \pi) \to B_3(0, r)$ is given as follows :

$$f(t, \theta, \varphi) = \begin{cases} t\cos(\theta)\sin(\varphi) \\ t\sin(\theta)\sin(\varphi) \\ t\cos(\varphi) \end{cases}$$

Compute the volume of $B_3(0, r)$, defined by

$$\int_{B_3(0,r)} dxdydz.$$