Only the exercises with an asterisk (*) will be corrected.

13.1. MC questions.

(a) Which of the following statements are true?

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a non-negative, continuous function. Then

☐ If $\lim_{x\to\infty} f(x) = 0$, the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite

☐ If the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite, then $\lim_{x\to\infty} f(x) = 0$

☐ If $\lim_{x\to\infty} f(x)$ does not exist, then the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite.

☐ If $\lim_{x\to\infty} f(x)$ exists and is nonzero, the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite.

(b) Let $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit ball with boundary $S := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f : D \to \mathbb{R}$ be a C^1 function, so that there exists a constant $c \neq 0$ for which

$$f(x, y) = c \quad \forall\ (x, y) \in S$$

holds. Choose the correct statement

☐ $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = c$

☐ $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = 0$

☐ The integral $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$ cannot be calculated with the given information.

*13.2. Integrals in $\mathbb{R}^2$.

Calculate the following integrals.

(a) $\displaystyle\int_D xy(x^2 + y^2)\,\mathrm{d}(x, y), \qquad D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 - 2x < 0\}.$

(b) $\displaystyle\int_D \frac{\sin(x^2 + y^2)}{2 + \cos(x^2 + y^2)}\,\mathrm{d}(x, y), \qquad D = \{(x, y) \in \mathbb{R}^2 \,|\, 1 < x^2 + y^2 < 4\}.$

(c) $\displaystyle\int_D \frac{1}{(x^2 + y^2)^2}\,\mathrm{d}(x, y), \qquad D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1, x + y > 1\}.$

13.3. The Astroid.

Let $a > 0$. The Astroid $A(a) \subset \mathbb{R}^2$ is the geometric figure in the plane defined by

$$A(a) := \left\{(x, y) \in \mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right\}$$

The construction of an Astroid is very geometric (see here). Let $B(a)$ denote the set

$$B(a) = \{(rx, ry) \subset \mathbb{R}^2 \mid r \in [0, 1],\ (x, y) \in A(a)\}.$$

Compute the area of $B(a)$ using the theorem of Green. Note that the theorem of Green is indeed applicable in this case.

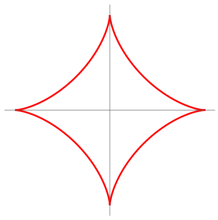

Figure 1: The red Astroid $A(a)$ is the boundary of $B(a)$.

*13.4. Improper integral.

Show that the limit

$$\lim_{R\to\infty} \int_{[0,R]^2} \sin(x^2 + y^2)\, d(x, y)$$

exists.

13.5. Fubini's theorem for an explicit integral.

By computing the integral (justify why it converges)

$$I = \int_{[0,\infty)\times[a,b]} e^{-xy} dxdy$$

in two different ways, where $0 < a < b$, compute

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$