14.1. MC questions.

(a) Which of the following is the tangent plane of the ellipsoid:

$$2x^2 + 2y^2 + \frac{1}{4}z^2 = 1$$

which is parallel to the plane $x + y + z = 1$?

- ☐ $x + y + z = 0$
- ☐ $x + y + z = k$ for $k \in \left\{\pm\frac{2}{\sqrt{5}}\right\}$
- ☐ $x + y + z = k$ for $k \in \left\{\pm\sqrt{5}\right\}$
- ☐ $x + y + z = k$ for $k \in \{\pm 1\}$

(b) For which functions $\phi, \psi, \chi\colon \{(x, y, z) \in \mathbb{R}^3 \mid y, z \neq 0\} \to \mathbb{R}$ there exists a function f such that $f_x = \phi$, $f_y = \psi$, and $f_z = \chi$?

- ☐ $\phi(x, y, z) = \frac{2x}{y},\ \psi(x, y, z) = 3y^2z^2 - \frac{x^2}{y^2},\ \chi(x, y, z) = 2y^3z.$
- ☐ $\phi(x, y, z) = e^y + 2xy^3z^2,\ \psi(x, y, z) = xe^y + 3x^2y^2z^2,\ \chi(x, y, z) = 2x^2y^3 + x.$
- ☐ $\phi(x, y, z) = e^z y\cos(xy),\ \psi(x, y, z) = e^z x\cos(xy),\ \chi(x, y, z) = e^z\sin(xy).$
- ☐ $\phi(x, y, z) = ze^x,\ \psi(x, y, z) = \frac{1}{z}\sin(\frac{y}{z}),\ \chi(x, y, z) = \frac{y}{z^2}\sin(\frac{y}{z}) + e^x.$

(c) For which of the following pairs of functions $\phi, \psi\colon \mathbb{R}^2 \to \mathbb{R}$ does there exist a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x, y) = \phi(x, y)$ and $f_y(x, y) = \psi(x, y)$?

- ☐ $\phi(x, y) = \left(\frac{y^3}{3}\right)\sinh(x) + y + \frac{x^2}{2}$ and $\psi(x, y) = y^2\cosh(x) + x.$
- ☐ $\phi(x, y) = x\sin(xy)$ and $\psi(x, y) = x\sin(xy) + 3x.$
- ☐ $\phi(x, y) = e^{x+\sin y}$ and $\psi(x, y) = \cos(y)e^{x+\sin(y)}.$
- ☐ $\phi(x, y) = xye^{x^2y}$ and $\psi(x, y) = \frac{1}{2}x^2e^{x^2y}.$
- ☐ $\phi(x, y) = \sinh(x^2y)$ and $\psi(x, y) = \sinh(xy^2).$

14.2. Extrema

Find the extrema of the function

$$f\colon D \to \mathbb{R}, \qquad f(x, y) = \exp(3y^2 - 1 - x^2)$$

for

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 4\right\}.$$

14.3. Volume

Calculate the volume between the ellipse $x^2 + 4y^2 \leq 1$ and the surface $z = 1 - x^2$.

Hint: Find the coordinates in the xy-plane in which the ellipse has a simpler form.

14.4. More volume

Let D be a surface bounded by the straight line from $(0, 0)$ to $(1, 0)$ and the arc parametrised by $\rho = \sin\left(\frac{\varphi}{4}\right)$, $0 \leq \varphi \leq 2\pi$. Calculate the volume of unit ball

$$\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

that lies over the surface D.

14.5. Intersection point

The equation $z = 2y^2 + x^2$ describes a surface S in $\mathbb{R}^3$, which contains the point $P = (1, 1, 3)$. Find the coordinates of the other point of S that lies on the normal to S at P.

14.6. Global extrema

Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 3x^2 - 2xy + 3y^2 - 4x - 4y + 4.$$

Find the global extrema of f on

$$B = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

14.7. Initial value problem Find the solution of the initial value problem:

$$u'' + u = F(t), \qquad u(0) = u'(0) = 0,$$

where F_0 is a constant and

$$F(t) = \begin{cases} F_0 t, & 0 \leq t < \pi \\ F_0(2\pi - t), & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi. \end{cases}$$

14.8. Tangent planes

Let f be any differentiable function of one variable. Show that all tangent planes of the surface

$$z = y \cdot f\left(\frac{x}{y}\right)$$

pass through the point $(0, 0, 0)$.

14.9. Global extrema

Let

$$f(x, y) \coloneqq xy(2x - 5y)$$

be a function defined on the closed square with corners at $(0, 0), (0, 2), (2, 2), (2, 0)$. Find the global extrema of f.