Only the exercises with an asterisk (*) will be corrected.

1.1. MC questions.

(a) How many solutions does the ODE

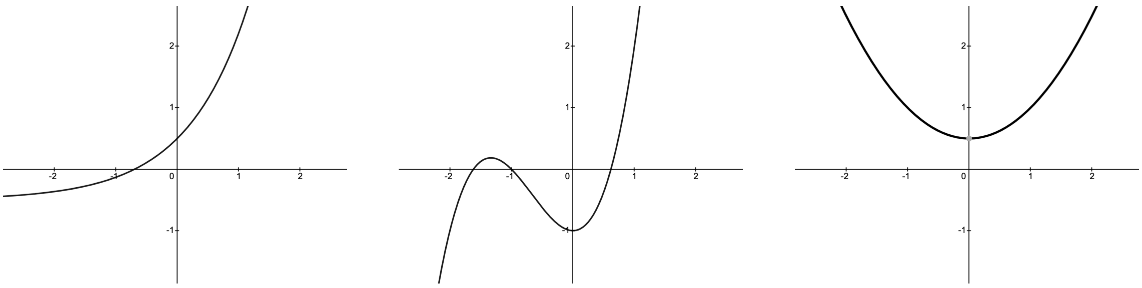
$$y'(x) = \frac{\sqrt{y(x)}}{1 - \operatorname{sgn}(y(x))}$$

have? Here, y is a function on $\mathbb{R}$ with values in $\mathbb{R}$.

- ☐ 0;
- ☑ 1;
- ☐ Infinitely many.

Solution. If $y(x) \geq 0$ for some $x \in \mathbb{R}$, then the ODE is not well defined, since $1 - \operatorname{sgn}(y(x)) = 0$. If $y(x) < 0$ for some $x \in \mathbb{R}$, then the ODE is again not well defined because $\sqrt{y(x)}$ is not well defined. If $y \equiv 0$, then the ODE is satisfied and $y \equiv 0$ is therefore the only solution.

(b) Which graph corresponds to $(x, g(x))$, where $g\colon \mathbb{R} \to \mathbb{R}$ is a solution to the ODE $y'(x) = x$?



(a) (b) (c)

- ☐ (a)
- ☐ (b)
- ☑ (c)

Solution. The solution to the given ODE is $y(x) = \frac{1}{2}x^2 + c$ for $c \in \mathbb{R}$. The graph of this function is shown in picture (c).

***1.2. ODE classification.** For each of the following expressions, determine whether they are an ODE, and if yes, determine their order and whether they are linear/nonlinear, homogeneous/inhomogeneous.

(a) $(y'(x)+1)^2 = y(x)+1$.

(b) $a\, x^3\, y'''(x) + \ x\, y'(x) = 1$, where $a \in \mathbb{R}$.

(c) $y'(x) - y(3x) = y(2x)$.

(d) $y(x) = xy'(x) + f(y'(x))$, where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

(e) $2xy'(x) + y(x) = e^x$.

Solution.

(a) Nonlinear ODE, first order.

(b) Linear inhomogeneous ODE. If $a \neq 0$, then it is of third order; if $a = 0$, it is of first order.

(c) Not an ODE for the presence of multiples of x in the arguments of y.

(d) It depends on f: if f is affine (i.e. $f(x) = ax + b$ for $a, b \in \mathbb{R}$), then the ODE is linear and of first order, inhomogeneous if $b \neq 0$ and homogeneous if $b = 0$. Otherwise it is nonlinear of first order.

(e) Linear inhomogeneous ODE of first order.

1.3. Verifying solutions. Consider the ODE:

$$y'' + y' - 6y = 0,$$

where $y\colon \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function.

(a) Verify that e^{-3x} and e^{2x} are solutions of the equation.

(b) Verify that $a\mathrm{e}^{-3x} + b\mathrm{e}^{2x}$ is again a solution of the equation for every $a, b \in \mathbb{R}$.

(c) Find all the (twice continuously differentiable) functions $\alpha : \mathbb{R} \to \mathbb{R}$ so that $\alpha(x)\mathrm{e}^{-3x} + \mathrm{e}^{2x}$ is a solution of the equation?

Hint: you may use the fact that the solutions to the equation $y'' - 5y' = 0$ are given by $y(x) = c_1 e^{5x} + c_2$ for $c_1, c_2 \in \mathbb{R}$.

Solution.

(a) & (b): recalling $\frac{d}{dx}(\mathrm{e}^{cx}) = c\mathrm{e}^{x}$ for $c \in \mathbb{R}$, it suffices to plug in the functions in the equation to verify the assertions.

(c) Calling $f(x) = \alpha(x)\mathrm{e}^{-3x} + \mathrm{e}^{2x}$, we have:

$$f'(x) = \alpha'(x)e^{-3x} - 3\alpha e^{-3x} + 2e^{2x},$$
$$f''(x) = \alpha''(x)e^{-3x} - 6\alpha'(x)e^{-3x} + 9\alpha(x)e^{-3x} + 4e^{2x},$$

hence

$$f''(x) + f'(x) - 6f(x) = \left(\alpha''(x) - 5\alpha'(x)\right)\mathrm{e}^{-3x},$$

and thus f solves the equation if and only if $\alpha'' - 5\alpha' = 0$. From the hint, it then follows that $\alpha(x) = c_1 e^{5x} + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$.

***1.4. Finding ODE with the given solution.** Find an ODE of the specified order solved by the given function:

(a) $\varphi(t) = \frac{1}{t}$, of 1st order,

(b) $\varphi(t) = t\cos t$, of 2nd order,

(c) $\varphi(t) = \mathrm{e}^{t^3}$, of 1st order.

Solution.

(a) Since $\varphi'(t) = -\frac{1}{t^2}$, one such ODE is $t\varphi'(t) + \varphi(t) = 0$.

(b) Since $\varphi'(t) = \cos t - t\sin t$ and $\varphi''(t) = -2\sin t - t\cos t$, one such ODE is $\varphi'' + \varphi = -2\sin t$.

(c) Since $\varphi'(t) = 3t^2 e^{t^3}$, one such ODE is $\varphi'(t) - 3t^2\varphi(t) = 0$.