Only the exercises with an asterisk (*) will be corrected.

2.1. MC questions.

(a) Which of the following functions has the property that its arc length is equal to its area under the curve?

For a function $f\colon \mathbb{R} \to \mathbb{R}$, the arc length between 0 and x is given by

$$\int_0^x \sqrt{1 + (f'(t))^2}\mathrm{d}t.$$

- (a) $f(x) = \cosh x$, ☑
- (b) $f(x) = x$ ☐
- (c) $f(x) = e^x$ ☐
- (d) $f(x) = \ln(x)$. ☐

Solution. The condition that the arc length of a function f is equal to its area under the curve can be written as:

$$\int_0^x \sqrt{1 + (f'(t))^2}\mathrm{d}t = \int_0^x f(t)\mathrm{d}t.$$

By differentiating both sides, we get:

$$f(x) = \sqrt{1 + (f'(x))^2}.$$

Substituting each of the given functions, we see that only (a) $f(x) = \cosh x$ is a solution of the above ODE.

(b) Choose the correct statement(s). Motivate your answers.

A particular solution of the ODE $y'' - 4y' - 12y = \sin(2x)$. is:

- (a) $\frac{1}{40}\cos(2x) - \frac{1}{40}\sin 2x$ ☐
- (b) $\frac{1}{15}\sin(2x)$ ☐
- (c) $\frac{1}{20}\cos(x)$ ☐
- (d) $\frac{1}{40}\cos(2x) - \frac{1}{20}\sin(2x)$. ☑

Solution. Directly plugging in the ODE each of the functions above, one sees that only (d) is correct.

***2.2. Motion of a spring.** A piece of mass m connected to a coil spring that can stretch along its length. If $k > 0$ denotes the spring constant, the equation of motion of such system is given, according to Hooke and Newton's laws, by

$$m\ddot{x}(t) = -kx(t), \qquad (\dagger)$$

where $x = x(t)$ denotes the position in time of the piece of mass along the vertical direction. Call $\omega = \sqrt{\frac{k}{m}}$. Find the solution of (†):

(a) with initial position $x(0) = 0$ and initial velocity $\dot{x}(0) = -\omega$.

(b) with initial position $x(0) = 1$ and position at time $t = \frac{\pi}{2\omega}$: $x(\frac{\pi}{2\omega}) = 3$.

(c) Is is possible to find a solution so that $x(t) \to 0$ as $t \to +\infty$?

Solution. Setting $\omega^2 = \frac{k}{m}$, we may rewrite the equation in normal form

$$\ddot{x} + \omega^2 x = 0.$$

The characteristic polynomial of the equation is $p(\lambda) = \lambda^2 + \omega^2$ whose roots are $\lambda_{1,2} = \pm i\omega$. Thus the general solution of the ODE is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \quad \text{for arbitrary } C_1, C_2 \in \mathbb{R}.$$

Consequently

(a) With $x(0) = 1$ and $\dot{x}(0) = 2\omega$ we get:

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1 \cos(0) + C_2 \sin(0) = C_1 = 0, \\ \dot{x}(0) = 2\omega \quad &\Rightarrow \quad -C_1\omega \sin(0) + C_2\omega \cos(0) = C_2\omega = -\omega. \end{aligned}$$

Hence it is $C_1 = 0$ and $C_2 = -1$. The required solution is then

$$x(t) = -\sin(\omega t).$$

(b) With $x(0) = 1$ and $x(\frac{\pi}{2\omega}) = 3$ we get:

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1 \cos(0) + C_2 \sin(0) = C_1 = 1, \\ x\left(\frac{\pi}{2\omega}\right) = 3 \quad &\Rightarrow \quad C_1 \cos\left(\frac{\pi}{2}\right) + C_2 \sin\left(\frac{\pi}{2}\right) = C_2 = 3. \end{aligned}$$

Hence it is $C_1 = 1$, $C_2 = 3$. The required solution is then

$$x(t) = \cos(\omega t) + 3\sin(\omega t).$$

(c) No, since every solution is periodic.

2.3. Projectile motion. Consider a ball being thrown from height $y_0 = 2\,\mathrm{m}$ at an angle $\theta = 30^\circ$ with initial speed $v = 15\,\mathrm{m\,s^{-1}}$. At which horizontal distance from its starting point will the ball hit the ground?

You may assume that ball's trajectory $(x(t), y(t))$, where $x\colon \mathbb{R}_+ \to \mathbb{R}$ is its horizontal coordinate and $y\colon \mathbb{R}_+ \to \mathbb{R}$ is its vertical coordinate, satisfies the following ODE:

$$\ddot{x} = 0, \qquad \ddot{y} = -g. \tag{$\clubsuit$}$$

Here, $g = 9.81\,\mathrm{m\,s^{-2}}$ is the gravitational acceleration.

Solution. By integrating the equations ($\clubsuit$) twice, we get

$$\begin{aligned} x(t) &= a_1 t + a_2, && a_1, a_2 \in \mathbb{R}, \\ y(t) &= -\frac{gt^2}{2} + b_1 t + b_2, && b_1, b_2 \in \mathbb{R}. \end{aligned}$$

As the initial conditions, we take:

$$\begin{aligned} x(0) &= a_2 = 0, & \dot{x}(0) &= a_1 = v_0 \cos\theta, \\ y(0) &= b_2 = y_0, & \dot{y}(0) &= b_1 = v_0 \sin\theta. \end{aligned}$$

and therefore the solutions are:

$$\begin{aligned} x(t) &= v_0 \cos\theta \cdot t, \\ y(t) &= -\frac{gt^2}{2} + v_0 \sin\theta \cdot t + y_0 \end{aligned}$$

The ball hits the ground when $y(t_{\text{hit}}) = 0$, i.e. at the time

$$t_{\text{hit}} = \frac{v_y + \sqrt{v_y^2 - 2gy_0}}{g}.$$

Note that the quadratic equation $y(t) = 0$ has two solutions, but only the positive one has physical sense.

The horizontal distance is then

$$x(t_{\text{hit}}) = v_x \cdot \frac{v_y + \sqrt{v_y^2 + 2gy_0}}{g} = 22.87\,\mathrm{m}.$$

***2.4. Radioactive decay.** Radioactive decay is the process by which an unstable atomic nucleus loses energy by radiation. It is described by the following equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N,$$

where $\lambda > 0$ is a positive constant and N is the amount of the radioactive material.

We define the half life T of a radioactive material as the time required for the half of the initial amount of radioactive material to decay.

Find the expression for the half life T in terms of the constant λ.

Solution. Let N_0 be the initial amount of radioactive material, i.e. $N(0) = N_0$. The solution to the ODE above is then

$$N(t) = N_0 e^{-\lambda t}.$$

We want to find the time T at which:

$$N_0 e^{-\lambda T} = \frac{1}{2} N_0.$$

From there it follows that

$$T = \frac{1}{\lambda} \ln 2.$$