Only the exercises with an asterisk (*) will be corrected.

3.1. MC questions

(a) We consider the ODE

$$u''(x) - 4u(x) = -1\,, \quad x \in (0,1)$$

with $u(0) = 0$ and $u'(1) + u(1) = 0$. What is the maximal value of $u(x)$?

- ☐ 0
- ☐ $\frac{\sqrt{3e^6+2e^4-e^2}}{e^4+1}$
- ☐ $e^6 + 2e^4 - e^2$
- ☑ $-\frac{1}{2}\frac{\sqrt{3e^6+2e^4-e^2}}{3e^4+1} + \frac{1}{4}$
- ☐ $-\frac{1}{4}\frac{\sqrt{3e^6+2e^4-e^2}}{e^4+1} + \frac{1}{2}$.

Solution. Solving the homogeneous problem: $u'' - 4u = 0$

$$p(\lambda) = \lambda^2 - 4 = 0 \quad \Rightarrow \; \lambda = \pm 2,$$

we get the solution $u_h(x) = c_1 e^{2x} + c_2 e^{-2x}$. As a guess for the particular solution, we take $u_p(x) = A$. Substituting this u_p into the ODE, we obtain $A = 1/4$. The general solution is thus given by:

$$u(x) \;=\; c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}$$

Substituting the boundary conditions, we get

$$\begin{aligned}
u(0) &= c_1 + c_2 + \frac{1}{4} = 0 \;\Rightarrow\; c_2 = -\frac{1}{4} - c_1 \\
u'(1) + u(1) &= 2c_1 e^2 - 2c_2 e^{-2} + c_1 e^2 + c_2 e^{-2} + \frac{1}{4} = 0 \\
\Rightarrow\; 0 &= 3c_1 e^2 - c_2 e^{-2} + \frac{1}{4} \\
\Rightarrow\; c_1(3e^2 + e^{-2}) &= -\frac{1}{4}(1 + e^{-2}) \\
\Rightarrow\; c_1 &= -\frac{1}{4}\frac{e^2+1}{3e^4+1} < 0 \\
\Rightarrow\; c_2 &= \frac{1}{4}\left(-1 + \frac{e^2+1}{3e^4+1}\right) = \frac{1}{4}\frac{e^2 - 3e^4}{3e^4+1} < 0
\end{aligned}$$

We thus get the solution:

$$u(x) = -\frac{1}{4}\frac{e^2+1}{3e^4+1}e^{2x} + \frac{1}{4}\frac{e^2-3e^4}{3e^4+1}e^{-2x} + \frac{1}{4}$$

The maximum is given by $u'(x) = 0$:

$$\begin{aligned} u'(x_0) &= 0 \Leftrightarrow e^{4x_0} = \frac{c_2}{c_1} \Leftrightarrow x_0 = \frac{1}{4}\log\frac{c_2}{c_1} = \frac{1}{4}\log\frac{3e^4-e^2}{e^2+1} = 0.7314 \in (0,1) \\ u''(x) &= 4c_1e^{2x} + 4c_2e^{-2x} < 0 \text{ on } (0,1) \Rightarrow x_0 \text{ is a maximum.} \\ u(x_0) &= -\frac{1}{2}\frac{\sqrt{3e^6+2e^4-e^2}}{3e^4+1} + \frac{1}{4} \end{aligned}$$

(b) Consider the following differential equation

$$y^{(4)} + 2y^{(2)} = 0. \qquad (\spadesuit)$$

Which of the following statements are true?

☑ The space of solutions of ($\spadesuit$) satisfying $y(0) = 0$ is a 3-dimensional vector space.

☐ The space of solutions of ($\spadesuit$) satisfying $y(0) = 1$ is a 3-dimensional vector space.

☐ The space of solutions of ($\spadesuit$) satisfying $\lim_{t\to\infty} y(t) = 0$ is a 3-dimensional vector space.

☐ The space of solutions of ($\spadesuit$) satisfying $y'(0) = 0$ is a 2-dimensional vector space.

Solution. The characteristic polynomial is $\lambda^4 + 2\lambda^2 = (\lambda^2+2)\lambda^2$. The zeros are $\lambda_1 = i\sqrt{2}, \lambda_2 = -i\sqrt{2}, \lambda_3 = \lambda_4 = 0$ and therefore the general solution is (without additional constraints) $C_1e^{i\sqrt{2}t} + C_2e^{-i\sqrt{2}t} + C_3 + C_4t$, and this gives us a 4-dimensional vector space.

The condition $y(0) = 0 \Leftrightarrow C_1 + C_2 + C_3 = 0$ reduces the space of solutions to a 3-dimensional vector space.

The condition $y(0) = 1 \Leftrightarrow C_1 + C_2 + C_3 = 1$ does not give us a vector space: if y_1 and y_2 are two solutions satisfying $y_1(0) = y_2(0) = 1$, then $y_1(0) + y_2(0) = 2 \neq 1$.

The condition $y(t) \to 0 (t \to \infty)$ is fulfilled if and only if $C_4 = 0$ (otherwise $y(t)$ would be unbounded), $C_1 = C_2 = 0$ (otherwise the solution would oscillate and the limit wouldn't exist) and therefore also $C_3 = 0$. This gives us a 0-dimensional vector space.

The condition $y'(0) = 0 \Leftrightarrow i\sqrt{2}C_1 - i\sqrt{2}C_2 + C_4 = 0$ gives us a 3-dimensional vector space.

3.2. A glance at systems. Let

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad t \in I \subseteq \mathbb{R} \tag{$*$}$$

be a linear, homogeneous ODE of order $n \geq 2$ with constant coefficients.

(a) Show that, by setting $z_1 = y$, $z_2 = y'$, ..., $z_n = y^{(n-1)}$, equation $(*)$ can be seen as a first-order *system* of ODEs:

$$\boldsymbol{z}' = A\boldsymbol{z}, \quad t \in I \tag{$\star$}$$

where $A \in M_{n\times n}(\mathbb{R})$ is a $n \times n$ matrix. Write down explicitly the expression for A.

(b) Prove that the characteristic polynomial of the ODE $(*)$ is the characteristic polynomial of the matrix A.

(c) Show that

$$\boldsymbol{\zeta}(t) = \mathrm{e}^{\lambda t}\boldsymbol{u}$$

is a solution of the homogeneous problem $(\star)$ if and only if λ is an eigenvalue of A and $\boldsymbol{u}$ is a corresponding eigenvector.

(d) *Fact:* Similarly as for ODE of order n, one can prove that for a system like $(\star)$, the set of solutions is vector space of dimension n.

Assuming that all the eigenvalues of A are distinct, use (c) and the fact above to find an explicit expression for the general solution of homogeneous system.

Solution.

(a) By construction $\boldsymbol{z}$ satisfies

$$\boldsymbol{z}' = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_{n-1}' \\ z_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix},$$

and thus

$$A = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathbb{I}_{n-1} & \\ 0 & & & \\ \hline -a_0 & -a_1 & \cdots & -a_{n-1} \end{array}\right),$$

where $\mathbb{I}_{n-1}$ denotes the $(n-1) \times (n-1)$ identity matrix.

(b) We proceed by induction on n. The assertion is trivial for $n = 1$. For the inductive step $(n-1) \to n$, we see that the characteristic polynomial of A is

$$p_A(\lambda) = \det(\lambda \mathbb{I}_n - A) = \det \begin{pmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \lambda & -1 \\ a_0 & a_1 & a_2 & & \cdots & \lambda + a_{n-1}, \end{pmatrix}$$

so choosing the first column to expand the expression we see that

$$p_A(\lambda) = \lambda \det \begin{pmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \lambda & -1 \\ a_1 & a_2 & a_3 & & \cdots & \lambda + a_{n-1}, \end{pmatrix} + (-1)^{2(n-1)} a_0,$$

and, by inductive hypothesis, the determinant in the expression above is $\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1$. In other words

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + \lambda a_1 + a_0.$$

(c) We have $\boldsymbol{\zeta}'(t) = \lambda e^{\lambda t} \boldsymbol{u}$, thus since $e^{\lambda t}$ is never zero,

$$\boldsymbol{\zeta}'(t) = A\boldsymbol{\zeta}(t) \quad \Longleftrightarrow \quad \lambda e^{\lambda t} \boldsymbol{u} = A[e^{\lambda t} \boldsymbol{u}] \quad \Longleftrightarrow \quad \lambda \boldsymbol{u} = A\boldsymbol{u},$$

and the last equality is precisely what defines eigenvalues and eigenvectors.

(d) From linear algebra, we know that, for two distinct eigenvalues, two (nonzero) corresponding eigenvectors are linearly independent. So, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are corresponding nonzero eigenvectors, the general solution for $(\star)$ has the form

$$c_1 e^{\lambda_1 t} \boldsymbol{u}_1 + c_2 e^{\lambda_2 t} \boldsymbol{u}_2 + \ldots + c_n e^{\lambda_n t} \boldsymbol{u}_n,$$

for arbitrary constants $c_i \in \mathbb{R}$.

*3.3. Solving ODEs

(a) Determine the general solution of the ODE

$$y''' + 5y'' - y' - 5y = 0.$$

(b) Determine the general solution of the ODE

$$y'' + y' = 0$$

with initial condition $y(1) = y'(1) = 2$.

(c) The ODE

$$f'' + 2qf' + (q + q^2)f = 0$$

contains a real parameter q. For which values of q do **all** the solutions remain bounded for $x \to \infty$?
Hint: Distinguish carefully the different cases concerning the zeros of the characteristic polynomial!

(d) Determine the general (real) solution of the ODE

$$y^{(4)} + 2y^{(2)} + y = 0,$$

which satisfies the initial conditions

$$y(0) = y'(0) = y''(0) = 0, \qquad y^{(3)}(0) = 1\,.$$

Solution.

(a) For the characteristic polynomial it holds true

$$\operatorname{chp}(\lambda) = \lambda^3 + 5\lambda^2 - \lambda - 5 \overset{!}{=} 0\,,$$

i.e.

$$\operatorname{chp}(\lambda) = (\lambda - 1) \cdot (\lambda^2 + 6\lambda + 5) = (\lambda - 1) \cdot (\lambda + 5) \cdot (\lambda + 1)$$

The zeros are

$$\lambda_1 = 1, \quad \lambda_2 = -5, \quad \lambda_3 = -1.$$

We obtain the general solution (all zeros are different and real)

$$y(x) = c_1 \cdot e^x + c_2 \cdot e^{-x} + c_3 \cdot e^{-5x}.$$

(b) For the equation $y'' + y' = 0$ the characteristic polynomial is

$$\operatorname{chp}(\lambda) = \lambda^2 + \lambda = 0 \Leftrightarrow \lambda_1 = 0, \quad \lambda_2 = -1$$

and hence the general solution is

$$y(x) = c_1 + c_2 \cdot e^{-x}$$

We insert the initial condition and get

$$y(1) = c_1 + c_2 \cdot e^{-1} \overset{!}{=} 2 \text{ and } y'(1) = -c_2 e^{-1} \overset{!}{=} 2$$

Therefore, $c_2 = -2e$ and $c_1 = 2 + 2 \cdot e \cdot e^{-1} = 4$, i.e. the solution is

$$y(x) = 4 - 2 \cdot e \cdot e^{-x} = 4 - 2 \cdot e^{1-x}.$$

(c) For the ODE

$$y'' + 2qy' + (q + q^2)y = 0$$

the characteristic equation is

$$\operatorname{chp}(\lambda) = \lambda^2 + 2q\lambda + (q + q^2) = 0$$

and hence the roots are

$$\lambda_{1,2} = \frac{-2q \pm \sqrt{4q^2 - 4(q + q^2)}}{2} = -q \pm \sqrt{-q}.$$

We classificate the roots according to the sign of q.
<u>$q > 0$:</u>
In this case, the zeros are not real,

$$\lambda_{1,2} = -q \pm i\sqrt{q}$$

and the general solution is

$$y(x) = e^{-q \cdot x} \cdot (c_1 \cos(\sqrt{q}x) + c_2 \sin(\sqrt{q}x)),$$

i.e. the solutions are all bounded for $q > 0$ when $x \to \infty$ (they tend to zero!).
<u>$q = 0$:</u>
The zeros are $\lambda_1 = \lambda_2 = 0$ and hence the general solution is

$$y(x) = c_1 + c_2 \cdot x$$

and it is bounded when $x \to \infty$ only when $c_2 = 0$, otherwise it is not bounded.
$\underline{q < 0}$: The zeros are real and the general solution is given by

$$y(x) = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x}.$$

Furthermore, we have that $\lambda_1 = -q + \sqrt{-q} > 0$ and so the solution with $c_1 \neq 0$ are surely not bounded.
In conclusion, we have that only for $q > 0$ all the solutions are bounded.

(d) The characteristic polynomial of the equation

$$y^{(4)} + 2y'' + y = 0$$

is

$$\mathrm{chp}(\lambda) = \lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = (\lambda - i)^2 \cdot (\lambda + i)^2,$$

whose roots are

$$\lambda_1 = \lambda_2 = i, \quad \text{and} \quad \lambda_3 = \lambda_4 = -i$$

Hence the general solution is given by a linear combination of the functions

$$e^{ix}, \quad xe^{ix}, \quad e^{-ix}, \quad xe^{-ix}$$

or equivalently by (since y must be a real function and $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$) a linear combination of the functions

$$\sin(x), \quad x\sin(x), \quad \cos(x), \quad x\cos(x),$$

i.e.

$$y(x) = c_1 \cdot \sin(x) + c_2 x \cdot \sin(x) + c_3 \cdot \cos(x) + c_4 \cdot x\cos(x)\ .$$

We must compute the derivatives of the general solution $y(x)$ and use the initial conditions in order to determine the constants.
We have

$$y(0) = c_3 \stackrel{!}{=} 0,$$

i.e. we now obtain

$$y(x) = c_1 \cdot \sin(x) + c_2 x \cdot \sin(x) + c_4 \cdot x\cos(x)$$

Hence

$$y'(x) = c_1 \cos(x) + c_2 \cdot (\sin(x) + x \cdot \cos(x)) + c_4(-x\sin(x) + \cos(x))$$

and

$$y'(0) = c_1 + c_4 \stackrel{!}{=} 0.$$

For the second derivative we get

$$y''(x) = -c_1 \sin(x) + c_2(-x\sin(x) + 2\cos(x)) + c_4(-2\sin(x) - x\cos(x))$$

and hence

$$y''(0) = 2c_2 \stackrel{!}{=} 0$$

and $c_2 = 0$.
For the third derivative we obtain

$$y'''(x) = -c_1 \cos(x) + c_4(-3\cos(x) + x \cdot \sin(x))$$

and

$$y'''(0) = -c_1 - 3c_4 \stackrel{!}{=} 1.$$

So we have for the last two constants c_1 and c_4 the system

$$c_1 + c_4 = 0 \text{ and } -c_1 - 3c_4 = 1 \Leftrightarrow c_1 = 1/2 \text{ and } c_4 = -1/2.$$

Then the solution is given by

$$y(x) = \frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x).$$

*3.4. ODE with given solutions.

(a) Find a linear ODE with constant coefficients such that e^{-x}, e^{x}, $e^{-\pi x}$ and $e^{\pi x}$ are solutions of the equation.

(b) Find a linear ODE with constant coefficients such that e^{-10x} and $e^{3x}\cos(3x)$ are solutions of the equation.

Solution.

(a) The characteristic polynomail must have as zeros -1, 1, π und $-\pi$. This implies

$$(\lambda + 1)(\lambda - 1)(\lambda + \pi)(\lambda - \pi) = \lambda^4 - \lambda^2(\pi^2 + 1) + \pi^2.$$

Hence, the equation is given by

$$y^{(4)}(x) - y''(x)(\pi^2 + 1) + \pi^2 y(x) = 0.$$

(b) The solution e^{-10x} originates from the root -10 of the characteristic polynomial, and $e^{3x}\cos(3x)$ from the complex conjugate roots $3+3i$ and $3-3i$. Thus, we obtain the characteristic polynomial

$$(\lambda+10)(\lambda-3-3i)(\lambda-3+3i)=\lambda^3+4\lambda^2-42\lambda+180.$$

This gives us the equation

$$y'''(x)+4y''(x)-42y'(x)+180y(x)=0.$$