Only the exercises with an asterisk (*) will be corrected.

4.1. MC questions.

(a) Which of the following guesses give us a particular solution to the following ODE:

$$y'' - 2y' + 2y = e^x \cos(x)?$$

- ☐ $y = ce^x \cos(x)$ for a constant $c \in \mathbb{R}$
- ☐ $y = cxe^x \cos(x)$ for a constant $c \in \mathbb{R}$
- ☑ $y = c_1 xe^x \cos(x) + c_2 xe^x \sin(x)$ for constants $c_1, c_2 \in \mathbb{R}$
- ☐ $y = c_1 e^{(1+i)x} + c_2 e^{(1-i)x}$ for constant $c_1, c_2 \in \mathbb{C}$
- ☑ $y = c_1 xe^{(1+i)x} + c_2 xe^{(1-i)x}$ for constant $c_1, c_2 \in \mathbb{C}$

Solution. The function on the right-hand side can be rewritten as

$$e^x \cos(x) = e^x \frac{e^{ix} + e^{-ix}}{2} = \frac{1}{2}\Big(e^{(1+i)x} + e^{(1-i)x}\Big).$$

As a guess for the solution, we take a linear combination of the guesses for $e^{(1+i)x}$ and $e^{(1-i)x}$. Since $1 \pm i$ are the zeros of the characteristic polynomial of the differential equation, we have to take the exponential function with the correct exponent and multiply it by x. We thus get:

$$y = c_1 xe^{(1+i)x} + c_2 xe^{(1-i)x}$$

with constants $c_1, c_2 \in \mathbb{C}$. Since $xe^{(1+i)x}$ and $xe^{(1-i)x}$ span the same subspace as $xe^x \cos(x)$ and $xe^x \sin(x)$, the following

$$y = \tilde{c_1} xe^x \cos(x) + \tilde{c_2} xe^x \sin(x)$$

for (different) constants $\tilde{c_1}, \tilde{c_2} \in \mathbb{C}$ is also a solution. If y is a particular solution, so is $\mathrm{Re}(y)$. We can thus take $\tilde{c_1}, \tilde{c_2} \in \mathbb{R}$.

(b) Which of the following statements are correct for $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^m \to \mathbb{R}$ and $x \in \mathbb{R}^n$?

- ☐ f is continuous at x if there exists a sequence $(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$, such that $f(x_k) \to f(x)$ for $k \to \infty$.

  Solution. False. This has to hold for every sequence that converges to x.

- ☐ The compositions $f \circ g$ and $g \circ f$ are always defined.

  Solution. False. The range of f is not the same as the domain of g when $n \neq m$.

- ☐ If $g \circ f$ and f are continuous, then g is also continuous.

  Solution. False. Take for example f constant. Then $g \circ f$ is also continuous, no matter how g is defined.

4.2. Inhomogeneous ODE.

Solve the following ODE:

$$y''' - 3y'' + 3y' - y = 4e^t.$$

Solution. The general solution of the homogeneous equation is given by:

$$y_{\text{h}}(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

for $c_1, c_2, c_3 \in \mathbb{R}$ constants.

We now look for a particular solution y_{p}. We look for an Ansatz of the form αe^t. Since e^t, te^t, and $t^2 e^t$ are solutions of the homogeneous equation, we must multiply our initial choice by t^3. Thus we take

$$y_{\text{p}}(t) = \alpha t^3 e^t.$$

Substituting this back into the original ODE and equating terms, we obtain $\alpha = \frac{2}{3}$.

The general solution is thus:

$$y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

***4.3. ODE change of variables.** Solve the following differential equation:

$$y' = (4x - y + 1)^2$$

Hint: use the substitution $u = 4x - y$.

Solution. From the change of variables, we obtain:

$$u' = 4 - y'$$

and thus the differential equation becomes

$$4 - u' = (u+1)^2.$$

After rearranging, we get:

$$u' = 4 - (u+1)^2 = (1-u)(3+u)$$

and after separating the variables:

$$\int \frac{1}{(1-u)(3+u)}\,\mathrm{d}u = x + \widetilde{C}.$$

The left-hand side can be rewritten as a sum of two integrals:

$$\frac{1}{4}\int \frac{1}{(1-u)}\,\mathrm{d}u + \frac{1}{4}\int \frac{1}{(3+u)}\,\mathrm{d}u = \frac{1}{4}\ln|3+u| - \frac{1}{4}\ln|1-u|.$$

We obtain

$$\ln\left|\frac{3+u}{1-u}\right| = 4x + 4\widetilde{C}$$

that is,

$$3 + u = Ce^{4x}(1-u) \qquad \Longleftrightarrow \qquad u = \frac{Ce^{4x} - 3}{1 + Ce^{4x}}.$$

The final result is thus

$$y(x) = 4x - \frac{Ce^{4x} - 3}{1 + Ce^{4x}}.$$

***4.4. (In)homogeneous ODEs.** Determine the general (real) solutions of the following differential equations, for $x > 0$.

(a) $x^2 y''(x) - 3xy'(x) + 5y(x) = 0.$

(b) $2xy'(x) - y(x) = \log(x).$

(c) $2y'' + 3y' + 10y = \sin(2x) + 1.$

Hint: For parts (a) and (b), consider the substitution $h(t) := y(e^t)$. If y is a solution to an equation above, then h solves a linear ODE with constant coefficients.

Solution. The substitution $x = e^t$ and $h(t) = y(e^t)$ give us

$$h(t) = y(e^t) = y(x), \qquad h'(x) = y'(e^t)e^t = xy'(x)$$

$$h''(t) = y''(e^t)e^{2t} + y'(e^t)e^t = x^2y''(x) + xy'(x).$$

We then get the following relations:

$$x^2y''(x) = h''(t) - h'(t), \qquad xy'(x) = h'(t), \qquad y(x) = h(t).$$

(a) The equation is equivalent to

$$h''(t) - 4h(t) + 5h(t) = 0.$$

The characteristic polynomial $\lambda^2 - 4\lambda + 5$ has $2 \pm \mathbf{i}$ as zeros ad thus the general solutions are

$$h(t) = c_1e^{2t}\cos(t) + c_2e^{2t}\sin(t)$$

and

$$y(x) = h(\log(x)) = c_1x^2\cos(\log(x)) + c_2x^2\sin(\log(x)).$$

(b) The equations are equivalent to

$$2h'(t) - h(t) = t.$$

The solution of the homogeneous equation $2h'(t) - h(t) = 0$ is given by $h_{hom}(t) = c_1e^{\frac{t}{2}}$. A particular solution is given by $h_p(t) = -(2+t)$ and thus the general solution is

$$h(t) = h_{hom}(t) + h_p(t) = c_1e^{\frac{t}{2}} - 2 - t.$$

We thus obtain:

$$y(x) = h(\log(x)) = c_1\sqrt{x} - 2 - \log(x).$$

(c) We consider the equations

$$2y'' + 3y' + 10y = \sin(2x) \tag{1}$$

and

$$2y'' + 3y' + 10y = 1. \tag{2}$$

We first solve the homogeneous problem

$$2y'' + 3y' + 10y = 0.$$

The characteristic polynomial is

$$\text{chp}(\lambda) = 2\lambda^2 + 3\lambda + 10$$

and has zeros

$$\lambda_1 = \frac{1}{4}\left(-3 + i\sqrt{71}\right) \quad \text{und} \quad \lambda_2 = \frac{1}{4}\left(-3 - i\sqrt{71}\right).$$

Therefore the real general solution $y_h(x)$ of the homogeneous problem (1) and (2) is

$$y_h(x) = e^{-\frac{3}{4}x}\left(C_1 \cos\left(\frac{\sqrt{71}}{4}x\right) + C_2 \sin\left(\frac{\sqrt{71}}{4}x\right)\right) \quad C_1, C_2 \in \mathbb{R}.$$

We need to find a particular solution for (1) and (2) For (1) we make a guess

$$y_{p_1}(x) = C_3 \sin(2x) + C_4 \cos(2x).$$

Explanation: $\sin(2x) = \frac{1}{2i}e^{2ix} - \frac{1}{2i}e^{-2ix}$. We thus take $A_0e^{2ix} + A_1e^{-2ix}$ as the guess for the particular solution. This can be written as $C_3\cos(2x) + C_4\sin(2x)$.

Substituting in (1), we get

$$(2C_3 - 6C_4)\sin(2x) + (2C_4 + 6C_3)\cos(2x) = \sin(2x)$$

and

$$\begin{aligned} 2C_3 - 6C_4 &= 1 \\ 2C_4 + 6C_3 &= 0 \end{aligned}$$

Wir find

$$C_3 = \frac{1}{20} \quad \text{und} \quad C_4 = -\frac{3}{20}.$$

Therefore

$$y_{p_1}(x) = \frac{1}{20}\sin(2x) - \frac{3}{20}\cos(2x)$$

is a particular solution for (1).

We now determine a particular solution to (2). We make a guess

$$y_{p_2}(x) = C_5.$$

By substituting, we calculate $C_5 = \frac{1}{10}$, and thus $y_{p_2}(x) = \frac{1}{10}$ is a solution of (2).

The general solution of

$$2y'' + 3y' + 10y = \sin(2x) + 1.$$

is therefore:

$$\begin{aligned} y(x) &= y_h(x) + y_{p_1}(x) + y_{p_2}(x) \\ &= e^{-\frac{3}{4}x}\left(C_1 \cos\left(\frac{\sqrt{71}}{4}x\right) + C_2 \sin\left(\frac{\sqrt{71}}{4}x\right)\right) + \frac{1}{20}\sin(2x) - \frac{3}{20}\cos(2x) + \frac{1}{10}. \end{aligned}$$