Only the exercises with an asterisk (*) will be corrected.

5.1. MC questions.

(a) Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to \mathbb{R}$ be given by $f(x, y) = \sin(y)^x$. Then for the partial derivatives of f it holds:

☐ $\frac{\partial f}{\partial x}(x, y) = x \sin(y)^{x-1}$ and $\frac{\partial f}{\partial y}(x, y) = \cos(y)^x$.

☑ $\frac{\partial f}{\partial x}(x, y) = \sin(y)^x \log(\sin(y))$ and $\frac{\partial f}{\partial y}(x, y) = x \cos(y) \sin(y)^{x-1}$.

Solution. We write $f(x, y) = e^{\log(\sin(y))x}$ and use the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= e^{\log(\sin(y))x} \frac{\partial f}{\partial x}(\log(\sin(y))x) \\ &= e^{\log(\sin(y))x} \log(\sin(y)) = \sin(y)^x \log(\sin(y)) \\ \frac{\partial f}{\partial y}(x, y) &= e^{\log(\sin(y))x} \frac{\partial}{\partial y}(\log(\sin(y))x) = \sin(y)^x \frac{x \cos(y)}{\sin(y)} \\ &= x \cos(y) \sin(y)^{x-1}.\end{aligned}$$

☐ $\frac{\partial f}{\partial x}(x, y) = \log(1 + |x|) \sin(y - x)$ and $\frac{\partial f}{\partial y}(x, y) = e^{\cos(y)}x$.

(b) On which set $M \subseteq \mathbb{R}^2$ does the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = |xy|$$

have both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

☑ $M = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ und } y \neq 0\} \cup \{(0, 0)\}$.

Solution. We first determine the set M_x of points (x, y) in which $\frac{\partial f}{\partial x}(x, y)$ exists. Since $f(x, y) = |x||y|$, these are exactly the points where the function $x \mapsto |x||y|$ is differentiable.

The function $x \mapsto |x|$ is differentiable in all points except $x = 0$ and thus for $y \neq 0$ the function $x \mapsto f(x, y)$ is differentiable in all points except $x = 0$. For $y = 0$ the function $x \mapsto f(x, 0) = 0$ is differentiable in all points. Therefore $M_x = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ und } y \neq 0\}$.

By symmetry, we obtain $M_x = \{(x, y) \in \mathbb{R}^2 \mid x = 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ und } y \neq 0\}$ and therefore it follows

$$M = M_x \cap M_y = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ und } y \neq 0\} \cup \{(0, 0)\}.$$

- ☐ $M = \mathbb{R}^2$.
- ☐ $M = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ und } y \neq 0\}$.

5.2. Limits in $\mathbb{R}^n$

(a) Let f be a function defined by $f(x,y) = \frac{y}{x-1}$ on the set $\{(x,y) \in \mathbb{R}^2 \mid x \neq 1\}$. Does the limit $\lim_{(x,y)\to(1,0)} f(x,y)$ exist? If it does, calculate it.

Solution. We see that f vanishes on the line $\{(x,y) \mid y = 0,\ x \neq 1\}$. If f had a limit when $(x,y) \to (1,0)$, then this limit would have to be equal to 0. However, the line $y = x - 1$ passes through $(1,0)$ and on this line f is equal to 1. Therefore $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist.

(b) Let f be a function defined by $f(x,y) = \frac{x^2 y}{x^4+y^2}$ on the set $\mathbb{R}^2 \backslash \{(0,0)\}$. Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If it does, calculate it.

Solution. We claim that the limit does not exist. The function f again vanishes on the line $\{(x,y) \mid y = 0,\ x \neq 0\}$. If the limit were to exist, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ would have to hold. But for $y = x^2$ we see that

$$f(x,x^2) = \frac{x^4}{x^4+x^4} = \frac{1}{2}.$$

Since the parabola $\{(x,x^2) \mid x \in \mathbb{R}\}$ passes through $(0,0)$, our claim follows.

(c) Let f be a function defined on the set $\{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ und } (x,y) \neq (1,0)\}$ by $f(x,y) = \frac{(x-1)^2 \ln(x)}{(x-1)^2+y^2}$ Does the limit $\lim_{(x,y)\to(1,0)} f(x,y)$ exist? If it does, calculate it.

Solution. We see that for (x,y) with $x > 0$ and $(x,y) \neq (1,0)$ the following holds:

$$|f(x,y)| = \frac{|(x-1)^2|}{|(x-1)^2+y^2|} |\ln(x)| \leq |\ln(x)|.$$

Since $\lim_{x\to1} \ln(x) = 0$, we see that $0 \leq \lim_{(x,y)\to(1,0)} |f(x,y)| \leq \lim_{x\to1} |\ln(x)| = 0$ and thus also $\lim_{(x,y)\to(1,0)} f(x,y) = 0$.

(d) Let f be a function defined by $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ on the set $\mathbb{R}^2 \backslash \{(0,0)\}$. Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If it does, calculate it.

Solution. We know that the function $\mathbb{R} \ni t \mapsto \sin(t)$ is continuously differentiable. We also know that

$$\frac{d}{dt}\Big|_{t=0} \sin(t) = \cos(0) = 1.$$

From the definition of the derivative, it follows

$$\sin(s) = \sin(s) - \sin(0) = \frac{d}{dt}|_{t=0}\sin(t)\cdot s + o(s) = s + o(s), \tag{1}$$

where $s \mapsto o(s)$ is a function which satisfies

$$\frac{o(s)}{s} \overset{s\to 0}{\longrightarrow} 0. \tag{2}$$

In polar coordinates we can write $(x, y) \neq (0, 0)$ as $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Since $x^2 + y^2 = r^2$, we see that $f(x, y) = f(r) = \frac{\sin(r^2)}{r^2}$ is independent of θ. We thus get $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} f(r)$ (if it exists). By (1) we have

$$f(r) = \frac{\sin(r^2)}{r^2} = \frac{r^2 + o(r^2)}{r^2} = 1 + \frac{o(r^2)}{r^2},$$

and (2) gives us

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0}\left(1 + \frac{o(r^2)}{r^2}\right) = 1,$$

and we are done.

*5.3. Continuity in $\mathbb{R}^n$

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$f(x, y) = \begin{cases} \frac{(x+y)y}{x^2+y^2} & \text{für } (x, y) \in \mathbb{R}^2\backslash\{(0, 0)\} \\ 0 & \text{für } (x, y) = (0, 0). \end{cases}$$

Is f continuous?

Solution. We claim that f is not continuous. It suffices to show that $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist. We see that f vanishes on the set $\{(x, y) \mid y = 0,\ x \neq 0\}$. If the limit of $f(x, y)$ for $(x, y) \to (0, 0)$ were to exist, it would have to be 0. On the other hand, we see that for $x = y$, it holds

$$f(x, x) = \frac{x^2 + y^2}{x^2 + y^2} = 1.$$

Since the line $\{(x, y) \in \mathbb{R}^2 \mid x = y\}$ passes through $(0, 0)$ our claim holds.

(b) Let $f(x,y) = \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ be defined on the set $\{(x,y) \in \mathbb{R}^2 \mid x,y > 0 \text{ und } x \neq y\}$. Can f be continuously extended to the set $\{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0\}$?

Solution. We see that for (x,y) with $x,y > 0$ and $x \neq y$ it holds:

$$f(x,y) = \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = x(\sqrt{x}+\sqrt{y}).$$

Thus f can be continuously extended to the set $\{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0\}$.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{für } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \\ 0 & \text{für } (x,y) = (0,0). \end{cases}$$

Is f continuous?

Solution. We claim that f is continuous. We have to show that $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$. We write $(x,y) \neq (0,0)$ in polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and obtain

$$\begin{aligned} x^2+y^2 &= r^2(\cos^2(\theta)+\sin^2(\theta)) = r^2 \\ x^3+y^3 &= r^3(\cos^3(\theta)+\sin^3(\theta)) \end{aligned}$$

and $|f(r,\theta)| = |r(\cos^3(\theta)+\sin^3(\theta))| \leq 2r$. Since $(x,y) \to (0,0)$, we have $r \downarrow 0$ and therefore

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{r\downarrow 0} |f(r,\theta)| = 0.$$

Our claim follows.

(d) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$g(x,y) = \begin{cases} x\sin(1/y), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

For which $(x,y) \in \mathbb{R}^2$ is g continuous?

Solution. We immediately see that g is continuous in (x,y) for $y \neq 0$. We also see that g is continuous in $(0,0)$: we have

$$0 \leq |g(x,y)| \leq |x| \overset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

Therefore it holds $\lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$ and our claim follows.

We now investigate what happens in the points $(x, 0)$ with $x \neq 0$. We claim that g is not continuous in such places. To show this, we find a sequence (x_n, y_n) such that $(x_n, y_n) \to (x, 0)$ for $n \to \infty$ but $g(x_n, y_n) \nrightarrow 0$ for $n \to \infty$. For example, we can choose $(x_n, y_n) = (x, \frac{2}{(4n+1)\pi})$. We thus see, that $(x_n, y_n) \to (x, 0)$ for $n \to \infty$, but

$$g(x_n, y_n) = x \sin(\tfrac{(4n+1)\pi}{2}) = x \sin(2n\pi + \tfrac{\pi}{2}) = x \sin(\tfrac{\pi}{2}) = x \neq 0$$

for all n. Our claim follows.

***5.4. Partial derivatives.** Calculate all the partial derivatives of the following functions:

(a) $f(x, y) = x$;

(b) $f(x, y) = e^{xy}$;

(c) $f(x, y) = x^y$;

(d) $f(x, y) = \frac{x-y}{x^2+y^2}$;

(e) $f(x, y) = x^2 y \sin(xy)$;

(f) $f(x, y, z) = xy^2 z^3$.

Solution.

(a)

$$\frac{\partial}{\partial x} f(x, y) = 1,$$
$$\frac{\partial}{\partial y} f(x, y) = 0$$

(b)

$$\frac{\partial}{\partial x} f(x, y) = y e^{xy},$$
$$\frac{\partial}{\partial y} f(x, y) = x e^{xy}$$

(c)

$$\frac{\partial}{\partial x} f(x, y) = y x^{y-1},$$
$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} e^{y \ln x} = \ln x e^{y \ln x} = x^y \ln x$$

(d)

$$\frac{\partial}{\partial x} f(x,y) = \frac{(x^2+y^2)-(x-y)2x}{(x^2+y^2)^2} = \frac{y^2-x^2+2xy}{(x^2+y^2)^2},$$
$$\frac{\partial}{\partial y} f(x,y) = \frac{-(x^2+y^2)-(x-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

(e)

$$\frac{\partial}{\partial x} f(x,y) = 2xy\sin(xy) + x^2y^2\cos(xy),$$
$$\frac{\partial}{\partial y} f(x,y) = x^2\sin(xy) + x^3y\cos(xy)$$

(f)

$$\frac{\partial}{\partial x} f(x,y,z) = y^2z^3,$$
$$\frac{\partial}{\partial y} f(x,y,z) = 2xyz^3,$$
$$\frac{\partial}{\partial z} f(x,y,z) = 3xy^2z^2$$